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BSDEs and Operator Deep BSDEs for dynamic risk evaluation

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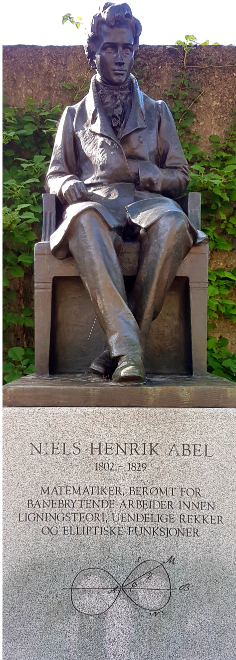
Based on works with **Emanuela Rosazza Gianin** (U. Milano Bicocca) and **Pere Diaz Lozano** (UiO)



1. Dynamic risk evaluation with attention to the time-scales

Time dependent phenomena require risk evaluation at different moments, leading to dynamic risk measurement.

We consider dynamic risk evaluation when risky positions in a portfolio have different time horizons.[1]

We identify two issues to discuss:

- **horizon risk**
- in a financial context, **interest rate uncertainty**

[1] D. diBartolomeo: Risk, Uncertainty and Time Horizon: What Most Risk Models Get Wrong! Newport Seminar, 2017.

Short on dynamic risk measures

A dynamic risk measure is a family of individual risk measures $(\rho_t)_{0 \leq t \leq T}$.

Namely, the mapping $\rho_t : \mathcal{X}_T \longrightarrow \mathcal{X}_t$ (where $\mathcal{X}_t$ represents $\mathcal{F}_t$-random variables) is:

1. monotone: if $X \leq Y$, then $\rho_t(X) \geq \rho_t(Y)$
2. cash additive: if $m \in \mathcal{X}_t$, then $\rho_t(X + m) = \rho_t(X) - m$
3. normalized: $\rho_t(0) = 0$
4. positive homogeneous: for $\lambda > 0$, then $\rho_t(\lambda X) = \lambda \rho_t(X)$
5. sub-additive: $\rho_t(X + Y) \leq \rho_t(X) + \rho_t(Y)$
6. **convex**: for $\lambda \in [0, 1]$, then $\rho_t\Big(\lambda X + (1 - \lambda) Y\Big) \leq \lambda \rho_t(X) + (1 - \lambda)\rho_t(Y)$

A **monetary evaluation** of an admissible risk: $\rho_t(X) = ess.inf\{m : \; m + X \in \mathcal{A}_t\}$.

Dynamic risk measures and BSDEs

The future uncertainty is here represented by a Brownian noise with the associated information flow. Random variables have moments and $\mathcal{X}_t = L^2(\mathcal{F}_t)$.

Characterisation of rm in terms of BSDEs.[2] Dynamic risk measures are associated to BSDEs:

$$Y_t = -X + \int_t^T g(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dB_s$$

The process $(Y_t)_t$ in the solution $(Y_t, Z_t)_{t\in[0,T]}$ is regarded as an operator depending characterised by the **driver** g and evaluated at $-X \in L^2(\mathcal{F}_T)$:

$$\rho_t(X) = Y_t, \qquad X \in L^2(\mathcal{F}_T),\ t \in [0,T].$$

See also the connection with *non-linear expectations.*

[2] See Peng (1997, 2003), Frittelli, Rosazza Gianin (2002, 2004), Rosazza Gianin (2006), ...

The properties of the driver g characterise the properties of $(\rho_t)_t$. For instance,

- When g is **Lipschitz**, existence and unicity of the solution are guaranteed.
- Beyond this case (e.g. quadratic), one can study concepts of "maximal solutions"[3].
- Among **quadratic drivers** we consider the cases of form

$$g(t, z, y) = a_t + f(y)|z|^2$$

  for which we have a unique solution for any terminal condition in $L^2(\mathcal{F}_T)$ [4]
- If $g(t, 0, 0) = 0$, then **normalisation** is guaranteed.
- Convexity of g in (y, z) provides a **convex** risk measures.
- When g does not depend on y, then **cash additive** is satisfied[5]

Other notices of interest

- The future may not be Gaussian, other family of noises considered[6]

[3] Kobilanski (2000) and also Barrieu and El Karoui (2009

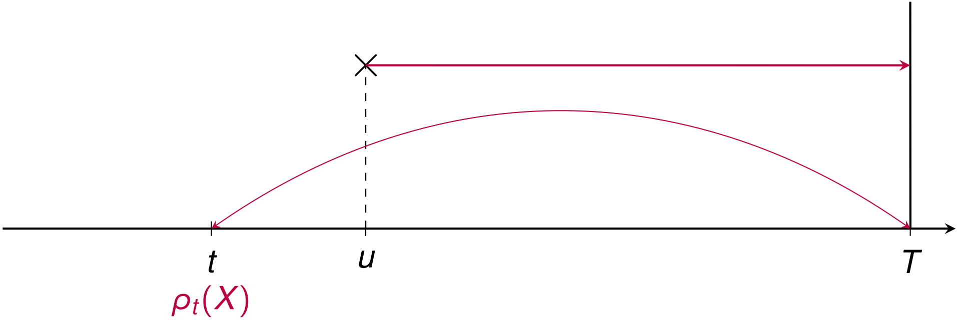
[4] Balhali et al. (2017).

[5] Barrieu and El Karoui (2009), Jiang (2008).

[6] Royer (2006), Quenez, Sulem (2013), Laeven, Stadje(2014, DiNunno, Sjursen (2014), Sulem, Øksendal (2019),...

Horizon risk and dynamic risk

Horizon risk emerges with the use of a wrong risk measure for the targeted horizon.



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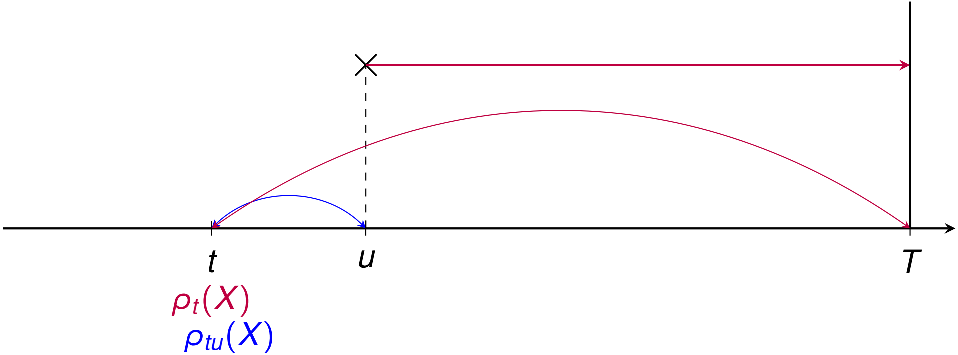

We then introduce fully dynamic rm and restriction property, and we quantify horizon risk by the h-longevity [7].

[7] Fully-dynamic risk measures: Bion-Nadal and DiNunno (2020). Horizon risk: DiNunno and Rosazza Gianin (2024)

2. Fully-dynamic risk measures

Fully-dynamic (convex) risk measure is a *family* $(\rho_{st})_{s,t}$ of risk measures :

$$\rho_{st} : \mathcal{X}_t \longrightarrow \mathcal{X}_s$$

In many applications, we consider each of the risk measure satisfying

- monotonicity, convexity
- cash additivity, i.e. for $X \in L^p(\mathcal{F}_t)$,

$$\rho_{st}(X + m) = \rho_{st}(X) - m, \text{ for all } m \in L^p(\mathcal{F}_s)$$

Comments

- We do not assume a priori that the risk measures ρ_{st} are normalised, i.e.

$$\rho_{st}(0) = 0, \quad \text{for all } s \le t,$$

- We do not assume that the risk measures have the restriction property, i.e.

$$\rho_{rs}(Y) = \rho_{rt}(Y), \qquad \text{for all } Y \in L^p(\mathcal{F}_s), \qquad r \le s \le t$$

Remark: relationship with dynamic risk measures
A *fully-dynamic risk measure with restriction property corresponds one-to-one with a dynamic risk measure*:

$$\rho_r(Y) = \rho_{rT}(Y) = \rho_{rs}(Y), \qquad \text{for all } Y \in L^p(\mathcal{F}_s), \qquad r \le s \le T.$$

3. Horizon risk and H-longevity

Once we drop restriction, we allow for the evaluation of horizon risk, and we introduce h-longevity as a penalisation for using a risk measure non-appropriate for the time window.

Definition. Horizon longevity or h-longevity is

$$\gamma(s,t,u,X) := \rho_{su}(X) - \rho_{st}(X) \geq 0$$

for any $0 \leq t \leq u, X \in \mathcal{X}_t$.

Cash additive BSDE approach

Consider a risk measures generated by BSDEs

$$\rho_{tu}(X) = Y_t = -X + \int_t^u g(r, Z_r)\,dr - \int_t^u Z_r\,dB_r, \quad X \in L^2(\mathcal{F}_u).$$

Proposition. The following properties are equivalent:

- $g(t, 0) = 0$ for any $t \in [0, T]$;
- each ρ_{st} is normalised;
- $(\rho_{st})_{s,t}$ satisfies the restriction property.

Hence, *to register horizon risk*, we need BSDEs with $g(t,0) \neq 0$. In fact,

Proposition. For a **Lipschitz** driver with $g(v,0) \geq 0$ for any v, h-longevity holds. Furthermore, with $\widetilde{Q}_X \sim P$ is a suitable probability measure depending on X, h-longevity is

$$\gamma(s,t,u,X) = E_{\widetilde{Q}_X}\left[\int_t^u g(v,0)dv \middle| \mathcal{F}_s\right].$$

Remark. **H-entropic type risk measures** derived from the quadratic BSDE

$$Y_t \quad = \quad -X + \int_t^u \left[b\frac{Z_r^2}{2} + a_r\right]dr - \int_t^u Z_r dB_r$$

with positive b and a_r. Then we have

$$\rho_{tu}(X) = \frac{1}{b}\ln\left(E_P\left[\exp\left(-bX + \int_t^u ba_r dr\right)\middle|\mathcal{F}_t\right]\right).$$

Hence, $(\rho_{st})_{s,t}$ is not normalized, does not satisfy the restriction property, but it shows has h-longevity.

4. LONG horizons, money, interest rates

Over long time horizons and in unstable times, the value of money cannot be neglected. So we have to embed the uncertainty on interest rates in risk evaluation.

Then we enter the domain of **cash subadditive risk measure**[8].

Risks are now expressed in unit of money where $€_t$ is the unit of money at time t.
Hence a financial investment at time t is denoted $X€_t$, where X is its volume.

Let $(D_{tu})_{0\leq t\leq u\leq T}$ be the family of discount factors D_{st} on the time interval $(t,u]$:

$$0 < d_{tu} \leq D_{tu}€_u \leq 1.$$

The unit of measurement for D_{tu} is $1/€_u$.

[8] El Karouui, Ravanelli (2009) and for the static case Filipovic (2008).

For any *cash additive* fully-dynamic risk measure $(\varphi_{tu})_{0\leq t\leq u\leq T}$ we define

$$\rho_{tu}(X) \triangleq \varphi_{tu}(D_{tu}X\text{€}_t), \qquad X \in L^p(\mathcal{F}_u).$$

Indeed, ρ_{tu} is **cash subadditive**. For any $X \in L^p(\mathcal{F}_u)$ and $m \in L^p_+(\mathcal{F}_t)$, we have

$$\begin{aligned}\rho_{tu}(X+m) &= \varphi_{tu}(D_{tu}(X+m)\text{€}_u) \\ &\geq \varphi_{tu}(D_{tu}X\text{€}_u + m) = \varphi_{tu}(D_{tu}X\text{€}_u) - m \\ &= \rho_{tu}(X) - m,\end{aligned}$$

thanks to monotonicity and $D_{tu}\text{€}_u \leq 1$.

Another cash subaddtive risk measure generated by the **ambiguity** of the interest rates is given by:

$$\mathcal{R}_{tu}(X) \triangleq \operatorname*{ess\,sup}_{0<d_{tu}\leq D_{tu}\text{€}_u\leq 1} \varphi_{tu}(D_{tu}X\text{€}_u)$$

In the framework of **cash non-additive** risk measures we can study h-longevity, normalisation, and time-consistency.

Cash non-additivity and BSDE

In a dynamic setting, we can generate cash non-additive risk measures from BSDEs with explicit dependence on Y in the driver:

$$Y_t = -X + \int_t^u g(s, Y_s, Z_s)\, ds - \int_t^u Z_s\, dB_s$$

- In particular, if $g(s, y, z)$ is decreasing in y for all (s, z), then ρ_{tu} is cash subadditive[9].

Proposition.

- ρ_{tu} has **restriction** if and only if $g(t, y, 0) = 0$ for a.a. t, y.
- If g Lipschitz or if $g(t, y, z) = a_t + f(y)|z|^2$ with f continuous, then ρ_{tu} is **normalised** if and only if $g(t, 0, 0) = 0$.

[9] See El Karoui, Ravanelli 2009)

Proposition for g Lipschitz

H-longevity holds if and only if $g(t, y, 0) \geq 0$ for any t, y. Furthermore, we have

$$\gamma(s,t,u,X) = E_{\widetilde{Q}_X}\left[e^{\int_s^u \Delta_y g(v)dv} \int_t^u g(v,-X,0)dv|\mathcal{F}_s\right], \quad s \leq t \leq u, X \in L^p(\mathcal{F}_t),$$

where $\widetilde{Q}_X \sim P$ is a suitable probability measure depending on X and

$$\Delta_y g(v) \triangleq \frac{g(v, Y_v^u, Z_v^u) - g(v, \bar{Y}_v^t, Z_v^u)}{Y_v^u - \bar{Y}_v^t} 1_{\{Y_v^t \neq \bar{Y}_v^u\}} \qquad \bar{Y}_v^t = \begin{cases} Y_v^t; & v \leq t \\ -X; & t < v \leq u \end{cases}$$

The probability $\widetilde{Q}_X$ can be interpreted as an **h-longevity premium measure.**[10]

[10] Di Nunno, Rosazza Gianin (2024).

5. hq-entropic risk measures on losses

Driven by considerations on capital requirements on potential losses in long term horizons, we consider

$$Y_t = X + \int_t^u \left[\frac{q}{2}\,\frac{Z_s^2}{1+(1-q)Y_s} + a(s)\right] ds - \int_t^T Z_s dB_s$$

The solution of such BSDE is Tsallis generalised entropy

$$Y_t = \ln_q E\Big[\exp_q\Big(X + \int_t^u a(s)ds\Big)\Big|\mathcal{F}_t\Big]$$

given in terms of the **generalised q-exponential** and **q-logarithmic functions** for $q \in (0,1)$:

$$\exp_q(x) = [1-(1-q)x]^{\frac{1}{1-q}}, \qquad x \geq -\frac{1}{1-q}$$

and

$$\ln_q(x) = \frac{x^{1-q}-1}{1-q}, \qquad x \geq 0$$

Definition. The **hq-entropic measure on losses** is then

$$\rho_{tu}^{hq}(X) = \ln_q E_P \left[\exp_q \left((X+\beta)^- + \alpha_q + \int_t^u a_s ds \right) \Big| \mathcal{F}_t \right],$$

here β represents a level of acceptable loss, while $\alpha_q \geq \frac{1}{q-1}$ and $q \in (0,1)$ provide how conservative the risk evaluation should be.

This risk measure is **convex**, **cash subadditive**, **NOT normalised**, **NOT restricted**, and there is **h-longevity** whenever $a_s > 0$.

Remark. Take $a \equiv 0$. For a fixed q, the higher is the value α_q, the more conservative is the corresponding measure $\rho_{tu}^q(X)$. In fact,

$$\alpha_q^1 \leq \alpha_q^2 \quad \Longrightarrow \quad \ln_q E\left[\exp_q((X+\beta)^- + \alpha_q^1) \big| \mathcal{F}_t\right] \leq \ln_q E\left[\exp_q((X+\beta)^- + \alpha_q^2) \big| \mathcal{F}_t\right].$$

Proposition. Take $a \equiv 0$. For any $X \in L^2(\mathcal{F}_u), \beta \in \mathbb{R}$, the q-entropic risk measure on losses ρ_{tu}^q is increasing in q and $\alpha_0 \leq \alpha_1 \leq \alpha_1$ and

$$E_P[(X+\beta)^- + \alpha_0 | \mathcal{F}_t] = \rho_{tu}^0(X) \leq \rho_{tu}^q(X) \leq \rho_{tu}^1(X) = \ln E_P[\exp(X+\beta)^- + \alpha_1 | \mathcal{F}_t].$$

6. Computational aspects: evaluation vs operator

- Classical numerical methods for BSDEs are established for the calculation of $\mathcal{E}^g(\xi|\mathcal{F}_t)$ for a fixed ξ.
- Instead, we will propose a numerical method to approximate the entire operator[11].

Indeed, approximations for a "fixed terminal condition" imply that whenever one needs to evaluate the solution at a different position, the algorithm needs to be re-executed from scratch.

The operator solution instead provides an immediate evaluation. This is particularly interesting when we wish to evaluate the risks with (path-wise) functional different types.

[11] Joint work with Diaz Lozano (2025)

Context.

We consider **Lispschitz type BSDEs** with g standard and any ξ terminal condition.

- Then the solution operators $(\mathcal{Y}, \mathcal{Z})$ are Lipschitz:

$$\mathcal{Y}: \quad \begin{array}{ccc} L^2(\mathcal{F}_T) & \to & \mathbb{H}_2(\mathbb{R}) \\ \xi & \mapsto & Y \end{array}$$

$$\mathcal{Z}: \quad \begin{array}{ccc} L^2(\mathcal{F}_T) & \to & \mathbb{H}_2(\mathbb{R}^d) \\ \xi & \mapsto & Z \end{array}$$

- Wiener chaos expansion. Any $\xi \in L^2(\mathcal{F}_T)$, admits representation

$$\xi = \sum_{a \in \mathcal{A}} d_a \times \prod_{i \geq 1} H_{a_i}\Big(\int_0^T h_i(s) \cdot dB_s\Big).$$

  - $\mathcal{A} = \{(a_1, a_2, \dots), a_i \in \mathbb{N} \cup \{0\}, \sum a_i < \infty\}$, i.e. in all the sequences only a finite number of terms does not vanish;
  - $(h_i)_{i \geq 1}$ orthonormal basis of $L^2([0, T]; \mathbb{R}^d)$ and H_n Hermite polynomial of order n.

- **Associated ∞-dim SDEs**: When considering

$$\xi = \sum_{a \in \mathcal{A}} d_a X_T^{(a_1, a_2, \dots)} \qquad \text{with} \qquad X_t^{(a_1, a_2, \dots)} := \prod_{i \geq 1} H_{a_i}\left(\int_0^t h_i(s) \cdot dB_s \right) \in \mathbb{R}^{\mathcal{A}},$$

We see that $X = (X_t)_t$ satisfies the ∞-dimensional linear SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t,$$

for some explicitly given coefficients.

- Then we connect the solution of the BSDE to the solution of the ∞-dim FBSDE system:

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$$

$$Y_t = \sum_{a \in \mathcal{A}} d_a X_T^a + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s \cdot dB_s.$$

- **Markovian structure.** For every $\xi \in L^2(\mathcal{F}_T)$, there exist $u, v \in L^2([0,T] \times \mathbb{R}^{\mathcal{A}}; \nu)$ such that

$$Y_t = u(t, X_t), \quad Z_t = v(t, X_t),$$

where ν is the pushforward finite measure of the law of $(X_t)_t$ on $[0,T] \times \mathbb{R}^{\mathcal{A}}$.

- This provides two ways of looking at the solution operators of the BSDE:

$$\mathcal{Y} : \begin{array}{ccc} L^2(\mathcal{F}_T) & \to & \mathbb{H}_2(\mathbb{R}) \\ \xi & \mapsto & Y \end{array}, \qquad \mathcal{Z} : \begin{array}{ccc} L^2(\mathcal{F}_T) & \to & \mathbb{H}_2(\mathbb{R}^d) \\ \xi & \mapsto & Z \end{array}$$

or

$$\mathcal{Y} : \begin{array}{ccc} L^2(\mathcal{F}_T) & \to & L^2([0,T] \times \mathbb{R}^{\mathcal{A}}, \nu) \\ \xi & \mapsto & u \end{array}, \qquad \mathcal{Z} : \begin{array}{ccc} L^2(\mathcal{F}_T) & \to & L^2([0,T] \times \mathbb{R}^{\mathcal{A}}, \nu) \\ \xi & \mapsto & v \end{array}.$$

equivalent to the functionals

$$\mathscr{Y}, \mathscr{Z} : \begin{array}{ccc} [0,T] \times \mathbb{R}^{\mathcal{A}} \times L^2(\mathcal{F}_T) & \longrightarrow & \mathbb{R} \times \mathbb{R}^d \\ (t, x, \xi) & \longmapsto & (\mathcal{Y}(\xi)(t,x), \mathcal{Z}(\xi)(t,x)) \end{array}$$

- **An Operator Euler scheme for BSDEs**

  Set $\pi := \{t_i\}_{0\le i\le n} \subset [0, T]$ and define recursively

$$\mathcal{Z}_i^\pi, \mathcal{Y}_i^\pi : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_{t_i}), \qquad i = n-1, \ldots, 0,$$

  by

$$\mathcal{Z}_i^\pi(\xi) := \frac{1}{\Delta t_i}\mathbb{E}_{t_i}\left[\mathcal{Y}_{i+1}^\pi(\xi)\Delta B_{t_i}\right], \quad \mathcal{Z}_n^\pi(\xi) := 0$$

$$\mathcal{Y}_i^\pi(\xi) := \mathbb{E}_{t_i}\left[\mathcal{Y}_{i+1}^\pi(\xi)\right] + \Delta t_i \times g(t_i, \mathcal{Y}_i^\pi(\xi), \mathcal{Z}_i^\pi(\xi)), \quad \mathcal{Y}_n^\pi(\xi) := \xi.$$

Convergence of the Operator Euler scheme for BSDEs

To measure the regularity of the solution operator of the BSDE, we define, for $\pi := \{t_i\}_{0\le i\le n} \subset [0, T]$, the functionals $(\mathcal{R}^Y, \mathcal{R}^Z)(\cdot, \pi)\colon L^2(\mathcal{F}_T) \to [0, \infty)$ by

$$\mathcal{R}^Y(\xi, \pi) := \max_{i=0,\dots,n-1} \mathbb{E}\Big[\sup_{t\in[t_i,t_{i+1}]} |\mathcal{Y}_t(\xi) - \mathcal{Y}_{t_i}(\xi)|^2 \Big]$$

and

$$\mathcal{R}^Z(\xi, \pi) := \mathbb{E}\Big[\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |\mathcal{Z}_t(\xi) - \overline{\mathcal{Z}}_i^{\pi}(\xi)|^2 dt \Big], \qquad \overline{\mathcal{Z}}_i^{\pi}(\xi) := \frac{1}{\Delta t_i} \mathbb{E}_{t_i}\Big[\int_{t_i}^{t_{i+1}} \mathcal{Z}_t(\xi) dt \Big].$$

Theorem: Convergence of the Operator Euler scheme for BSDEs.

Define the error as

$$\mathcal{E}(\xi,\pi) := \max_{0\le i\le n-1} \mathbb{E}\Big[\sup_{t\in[t_i,t_{i+1}]} |\mathcal{Y}_t(\xi)-\mathcal{Y}_i^\pi(\xi)|^2\Big] + \mathbb{E}\Big[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} |\mathcal{Z}_t(\xi)-\mathcal{Z}_i^\pi(\xi)|^2 dt\Big].$$

Then we have:

$$\mathcal{E}(\xi,\pi) \le C\Big(|\pi| + \mathcal{R}^Y(\xi,\pi) + \mathcal{R}^Z(\xi,\pi)\Big).$$

- **Pointwise convergence:** for each ξ, we have $\mathcal{E}(\xi,\pi)\to 0$ when $|\pi|\to 0$
- **Uniform convergence in compact subsets:** if $\pi_n \subset \pi_{n+1}$, then for all compact subsets $K\subset L^2(\mathcal{F}_T)$,

$$\sup_{\xi\in K}\mathcal{E}(\xi,\pi_n)\to 0 \quad \text{when } n\to\infty.$$

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$$\mathcal{R}^Y(\xi,\pi) := \max_{i=0,\dots,n-1} \mathbb{E}\Big[\sup_{t\in[t_i,t_{i+1}]} |\mathcal{Y}_t(\xi) - \mathcal{Y}_{t_i}(\xi)|^2\Big]$$

and

$$\mathcal{R}^Z(\xi,\pi) := \mathbb{E}\Big[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} |\mathcal{Z}_t(\xi) - \overline{\mathcal{Z}}_i^\pi(\xi)|^2 dt\Big], \qquad \overline{\mathcal{Z}}_i^\pi(\xi) := \frac{1}{\Delta t_i}\mathbb{E}_{t_i}\Big[\int_{t_i}^{t_{i+1}} \mathcal{Z}_t(\xi) dt\Big].$$

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Then we have:

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- **Uniform convergence in compact subsets:** if $\pi_n \subset \pi_{n+1}$, then for all compact subsets $K \subset L^2(\mathcal{F}_T)$,

$$\sup_{\xi\in K} \mathcal{E}(\xi,\pi_n) \to 0 \quad \text{when } n \to \infty.$$

Recalling the equivalence between the solution operators $(\mathcal{Y}, \mathcal{Z})$ and some measurable functionals $(\mathscr{Y}, \mathscr{Z})$, we can now express the approximations $(\mathcal{Y}_i^\pi, \mathcal{Z}_i^\pi)$ in terms of X_{t_i}.

We prove an equivalent version for the approximation $(\mathcal{Y}_i^\pi, \mathcal{Z}_i^\pi)$.

Theorem For $i = 0, \ldots, n-1$, there exist measurable functionals $\mathscr{Y}_i^\pi, \mathscr{Z}_i^\pi : \mathbb{R}^{\mathcal{A}} \times L^2(\mathcal{F}_T) \to \mathbb{R} \times \mathbb{R}^d$ such that

$$\mathcal{Y}_i^\pi(\xi) = \mathscr{Y}_i^\pi(X_{t_i}, \xi), \quad \mathcal{Z}_i^\pi(\xi) = \mathscr{Z}_i^\pi(X_{t_i}, \xi) \quad \mathbb{P} - a.s.$$

In the implementation, we will have to compute these functionals $\mathscr{Y}_i^\pi, \mathscr{Z}_i^\pi$ by approximations over a finite-dim domain.

Theorem: Convergence of the finite-dimensional approximation Let $p, M \in \mathbb{N}$. Then, for all $\xi \in L^2(\mathcal{F}_T)$, we have that

$$\max_{0 \leq i \leq n} \mathbb{E}\left|\mathscr{Y}_i^\pi(X_{t_i}, \xi) - (\mathscr{Y}_i^\pi \circ \Pi)(X_{t_i}, \xi)\right|^2 + \sum_{i=0}^{n-1} \mathbb{E}\Big[\int_{t_i}^{t_i+1} |\mathscr{Z}_i^\pi(X_{t_i}, \xi) - (\mathscr{Z}_i^\pi \circ \Pi)(X_{t_i}, \xi)|^2 dt\Big] \leq C\mathbb{E}|\xi - \Pi_{p,M}(\xi)|^2.$$

Comments

- Let $p \in \mathbb{N}$ denote the maximum order of the chaos decomposition, and $M \in \mathbb{N}$ the number of elements in the truncated basis of $L^2([0,T];\mathbb{R}^d)$. For $\xi \in L^2(\mathcal{F}_T)$, we define its projection

$$\Pi_{p,M}(\xi) := \sum_{k=0}^{p} \sum_{|a|=k} d_a X_T^a, \quad a = (a_1, \ldots, a_M). \tag{1}$$

  Recall that each $a = (a_1, \ldots, a_M)$ with $|a| = k$ means that $\sum_{j=1}^{M} a_j = k$.

  The dimension of $\Pi_{p,M}(L^2(\mathcal{F}_T))$ is finite.

- When we restrict the terminal condition ξ to $\Pi_{p,M}(L^2(\mathcal{F}_T))$, the ∞–dim FBSDE is equivalent to a finite-dimensional one, since the terminal condition only depends on a finite number of dimensions of the forward SDE.

7. Deep Operator BSDE - numerical example

Due to the high dimensionality of the projection, we choose **to use neural networks to approximate the maps** $(\mathscr{Y}_i^\pi \circ \Pi, \mathscr{Z}_i^\pi \circ \Pi)_{i=0,\dots,n-1}$.

We call such method the **Deep Operator BSDE**.

- We first fix the generator g, and train the method on 28 families of terminal conditions, each one with $1-3$ parameters.
- We approximate the coefficients of the truncated chaos expansion for several terminal conditions belonging to this subset, from which we keep the maximum and the minimum along each $a \in \mathcal{A}_{p,M}$ to find the coefficients and the compact subset for approximating the solution operator.
- Note that the method does not see the individual families during training, which underlines that the learned solution operator is genuinely defined on a much higher dimensional domain.

- We choose to model $(\mathscr{Y}_i^\pi, \mathscr{Z}_i^\pi)$ with the Multilevel Neural Networks presented in Julius Berner et. al. (2020)[12] . The number of parameters is around 2.8M.
- We choose $p = 3$ and $M = 5$. The time partition π for the Euler scheme is $\pi = \{i \times \frac{T}{10} : 0 \leq i \leq 10\}$.

  We set the dimension of the Brownian motion to $d = 2$. The number of chaos coefficients is 286.
- As for the baseline, we use the method proposed in Briand and Labart (2014)[13].

  NOTE that these baseline methods evaluate the solution operators at a particular terminal condition, while our proposed methodology learns the whole solution operator.

[12] J. Berner et. al. "Numerically solving parametric families of high-dimensional kolmogorov partial differential equations via deep learning", 2020

[13] P. Briand and C. Labart. "Simulation of BSDEs by Wiener Chaos Expansion". Annals Appl. Probab. 24.3 (2014), pp. 1129–1171.

Numerical results

We consider the case of pricing and hedging of options in a Black-Scholes setting of dimension 2. The dynamics are given by

$$S_t^j = s_0^j e^{(\mu^j - (\sigma^j)^2/2)t + \sigma^j B_t^j}, \quad \forall t \in [0, T], \quad j = \{1, 2\}, \quad \langle B^1, B^2 \rangle_t = \rho t.$$

If one assumes that the borrowing rate R is higher than the lending rate r, pricing and hedging an option ξ is equivalent to solving a BSDE with terminal condition ξ and nonlinear generator g defined by

$$g(t, y, z) = -ry - \theta \cdot z + (R - r)\Big(y - (\Sigma^{-1} z)_1 - (\Sigma^{-1} z)_2\Big)_-.$$

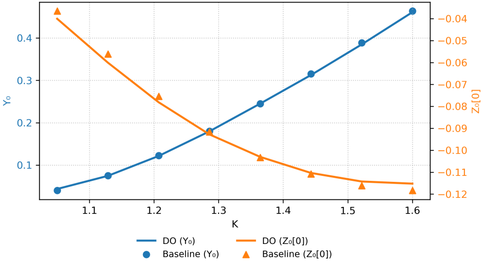
We assess the discrepancy between the Deep Operator BSDE and the baseline using the scaled relative error

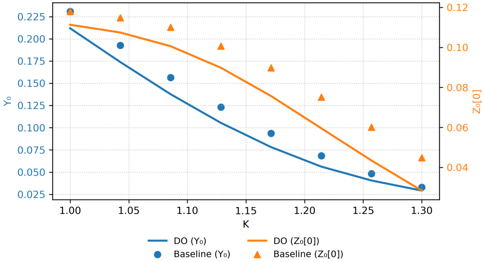
$$\mathrm{err}_Y(\xi) = \frac{|\mathcal{Y}_0^{\mathrm{DO}}(\xi) - \mathcal{Y}_0^{\mathrm{Base}}(\xi)|}{1 + |\mathcal{Y}_0^{\mathrm{Base}}(\xi)|},$$

with an analogous definition for Z.

Option	err_Y ($\times 10^{-3}$)	err_Z ($\times 10^{-2}$)
Call – Single	2.21 ± 0.65	0.41 ± 0.06
Call – Basket Weighted	2.58 ± 0.74	0.41 ± 0.07
Call – Spread	7.54 ± 1.08	0.25 ± 0.04
Call – Max	3.66 ± 0.42	0.58 ± 0.07
Call – Min	1.75 ± 0.98	0.32 ± 0.03
Call – Geometric	2.56 ± 0.73	0.41 ± 0.06
Call – Ratio	6.28 ± 1.02	0.25 ± 0.04
Put – Single	3.60 ± 0.16	0.24 ± 0.09
Put – Basket Weighted	3.19 ± 0.28	0.26 ± 0.09
Put – Spread	7.53 ± 1.09	0.27 ± 0.04
Put – Max	1.43 ± 0.38	0.18 ± 0.04
Put – Min	1.38 ± 0.49	0.38 ± 0.10
Put – Geometric	3.08 ± 0.27	0.26 ± 0.09
Put – Ratio	6.52 ± 1.02	0.35 ± 0.06
Asian Call – Single	11.00 ± 0.76	1.33 ± 0.07
Asian Put – Single	7.07 ± 1.39	0.20 ± 0.03
Asian Call – Basket Weighted	10.87 ± 0.80	1.21 ± 0.10
Asian Put – Basket Weighted	8.01 ± 1.35	0.18 ± 0.04
Asian Call – Spread	2.13 ± 0.66	1.01 ± 0.08
Asian Put – Spread	2.17 ± 0.64	1.00 ± 0.08
Asian Call – Max	13.06 ± 0.83	1.58 ± 0.08
Asian Put – Max	9.53 ± 1.16	0.30 ± 0.06
Asian Call – Min	3.22 ± 1.11	0.98 ± 0.09
Asian Put – Min	3.64 ± 1.14	0.28 ± 0.13

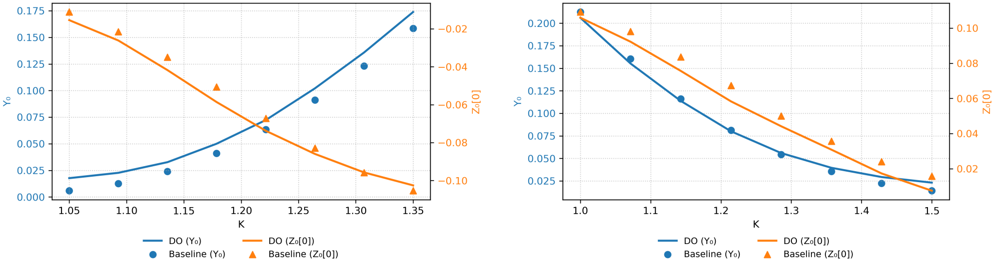
Table: Summary of some option families and their errors (mean $\pm$ std).





Left: Put – Max options, $\xi(K) = (K - \max(S_T^1, S_T^2))_+$, $K \in [1.05, 1.60]$.

Right: Asian Call – Max options, $\xi(K) = \left(\frac{1}{10}\sum_{j=1}^{10} \max(S_{t_j}^1, S_{t_j}^2) - K\right)_+$, $K \in [1.00, 1.30]$.



Left: Asian Put – Max options, $\xi(K) = \left(K - \frac{1}{10}\sum_{j=1}^{10} \max(S^1_{t_j}, S^2_{t_j})\right)_+, \quad K \in [1.05, 1.35]$.

Right: Call – Max options, $\xi(K) = (\max(S^1_T, S^2_T) - K)_+, \quad K \in [1.00, 1.50]$.

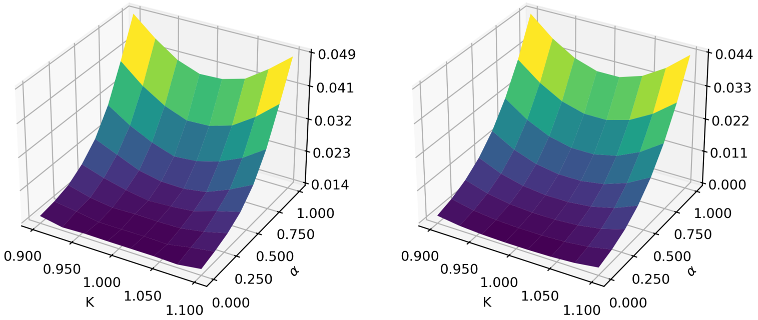


Figure: Comparison of Y_0.

Asian Put – Basket weighted options, $\xi(K,\alpha) = \left(K - \frac{1}{10}\sum_{j=1}^{10}(\alpha S^1_{t_j} + (1-\alpha)S^2_{t_j})\right)_+$,
$K \in [0.90, 1.10]$, $\alpha \in [0, 1]$.

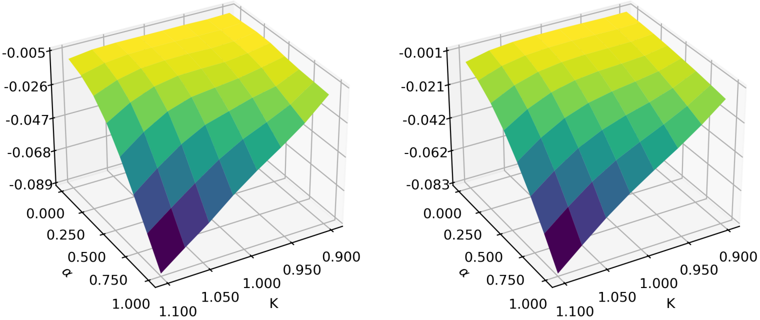


Figure: Comparison of Z_0.

Asian Put – Basket weighted options, $\xi(K,\alpha) = \left(K - \frac{1}{10}\sum_{j=1}^{10}(\alpha S^1_{t_j} + (1-\alpha)S^2_{t_j})\right)_+$, $K \in [0.90, 1.10],\ \alpha \in [0,1]$.

THANK YOU FOR YOUR ATTENTION

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