# Tipping point analysis of real-world complex systems

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National Physical Laboratory, UK
28 April 2026

# About NPL

The UK's national **metrological** laboratory

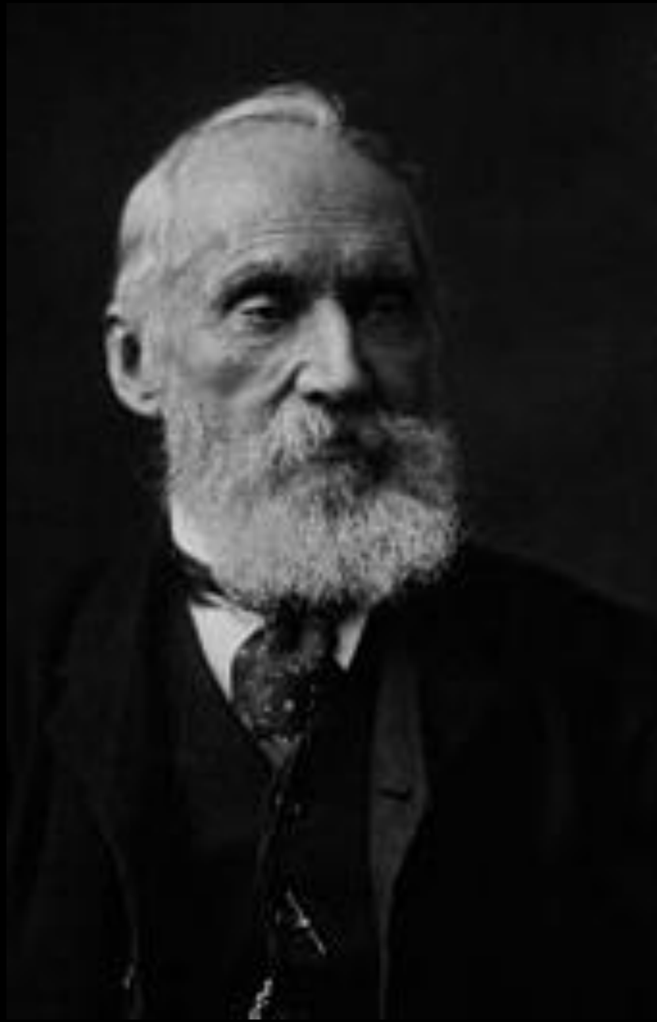
- Founded in **1900,** initially located in the Bushy House of the Bushy Park, Greater London
- World leading **National Measurement Institute**
- ~1200 staff; 550+ specialists in Measurement Science plus 200 visiting researchers pa
- State-of-the-art laboratory facilities
- 388 Laboratories (35,746 sq. metres)
- The heart of the UK's **National Measurement System** to support business and society
- Experts in **Knowledge Transfer**

# Metrology: Magna Carta (1215)

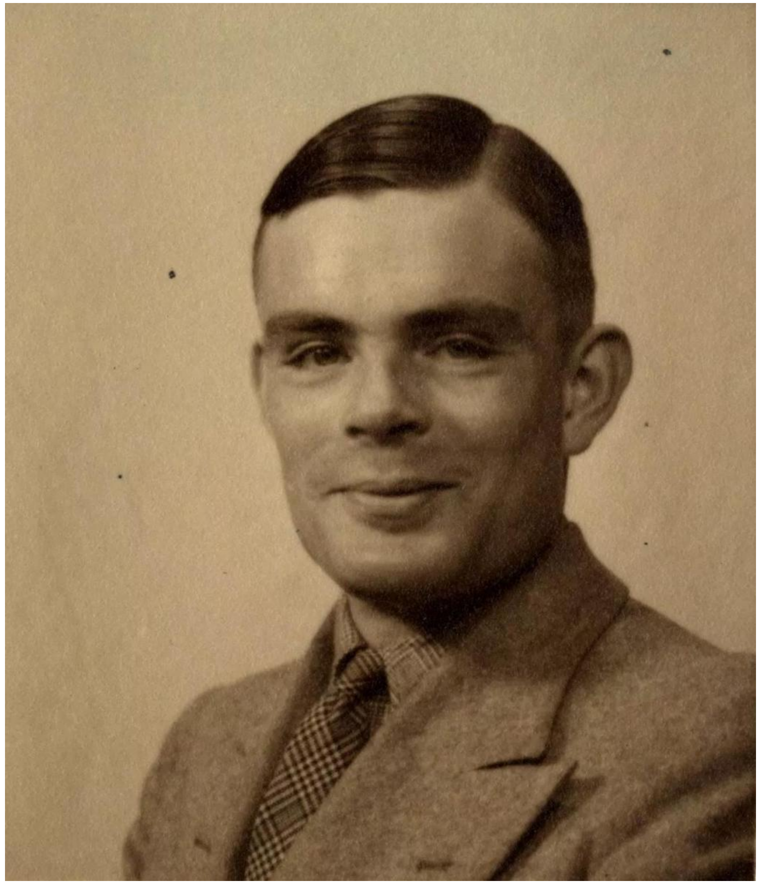
*"Let there be one measure of wine throughout our kingdom, and one measure of ale, and one measure of corn, namely the London quarter, and one width of cloth - whether dyed, russet or halberget, namely two ells within the selvedges; let it be the same with weights as with measures."*

“ When you can measure what you are speaking about and express it in numbers you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge of it is a meagre and unsatisfactory kind”

Lord Kelvin



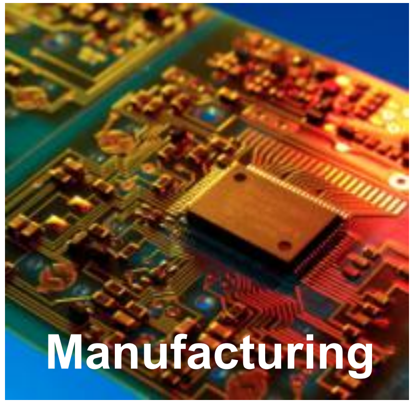
# Alan Turing (1912-1954)

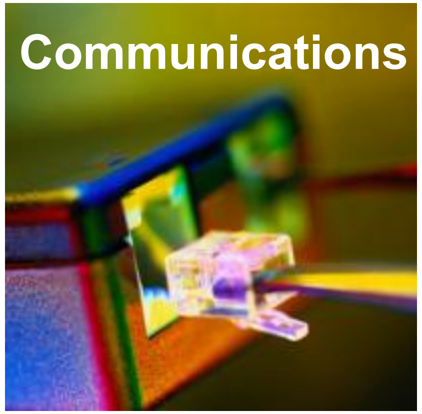
- Worked at NPL in 1945-1948
- Started developing the plan of the ACE machine that was later manufactured at the University of Manchester
- Paper “Intelligent Machinery” (1948)
- Lived near the Bushy Park

# The growing demand for better measurements, modelling and data analytics

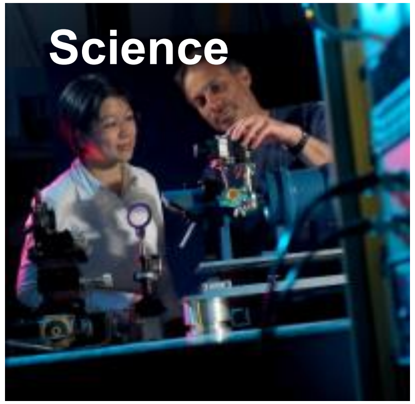











**2% of UK GDP dependent on a robust measurement system**

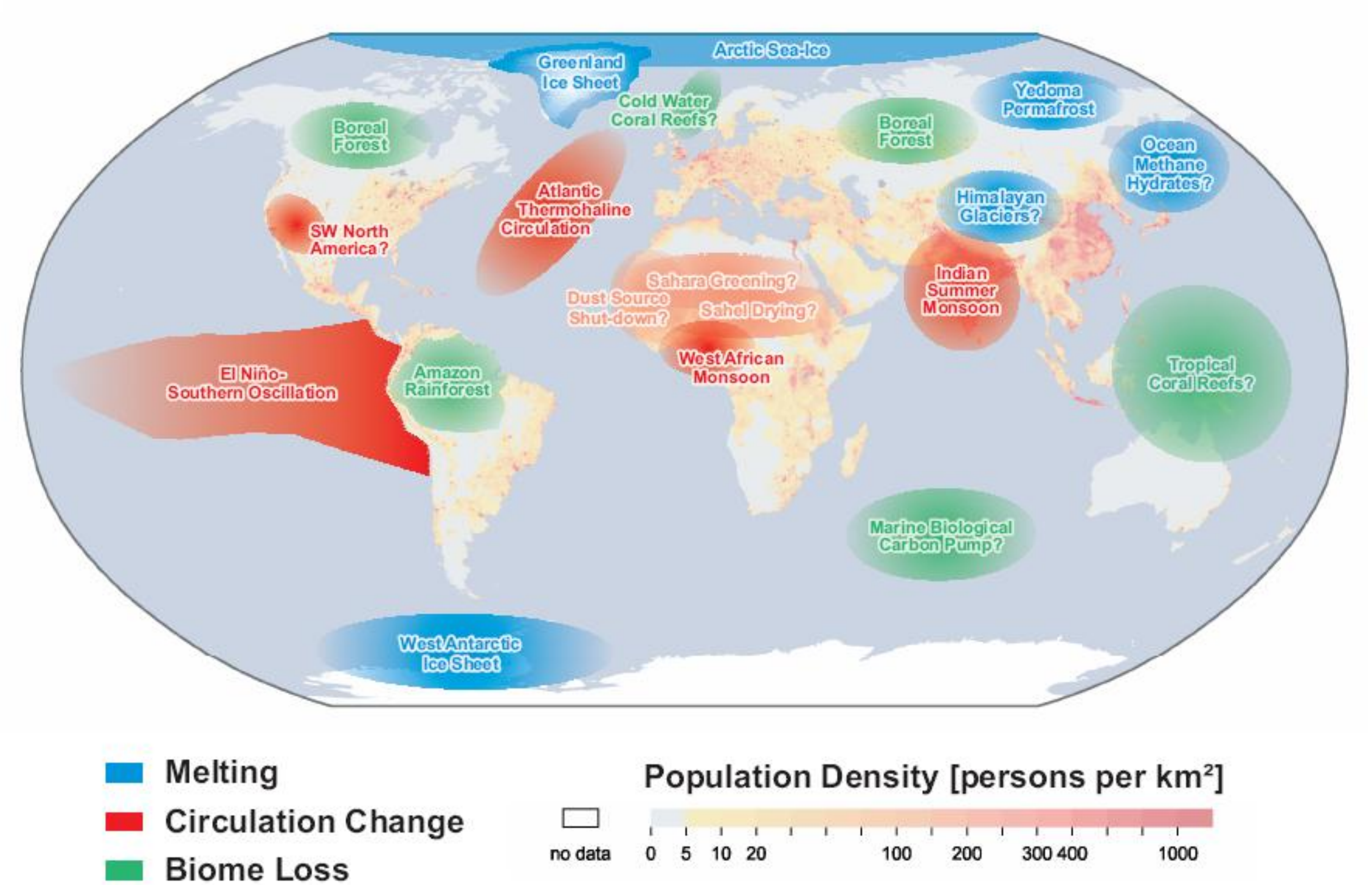
# Why study tipping points (bifurcations & transitions)? They are everywhere!

- Climatology (Greenland ice sheet melting)
- Thermoacoustics (work of Prof. Sujith)
- Semiconduntors (failure of components)
- Healthcare (epidemics)
- Cardiology (heart attacks)
- Power grids (shortcuts)
- Socio-physics (uptake of innovations)
- Ecology (species dynamics)
- Neuroscience (seizures)
- Structure health monitoring (failure of bridges)
- Building management systems (failure of ventilation)
- Chemical reactions (compounds concentrations)

# Tipping points: [subjective] timeline of development

- Bifurcation theory in XIX-XX centuries - analytical
- Lag-1 ACF: “degenerate fingerprinting” (Held & Kleinen 2004) – time series analysis
- DFA: “modified degenerate fingerprinting” (Livina & Lenton 2007) – detrending and scaling
- Tipping elements & tipping points (Lenton et al 2008)
- Potential model for climate data (Kwasniok & Lohmann 2009)
- Potential contour plot for tipping detection (Livina et al 2010)
- Potential forecasting (Livina et al 2013)
- Past 15 years: spatial fields, multivariate tipping, cascades, complex networks; applications in ecology, climatology, structure health monitoring, engineering systems

# Tipping elements in the climate system




Lenton et al. PNAS, 2008

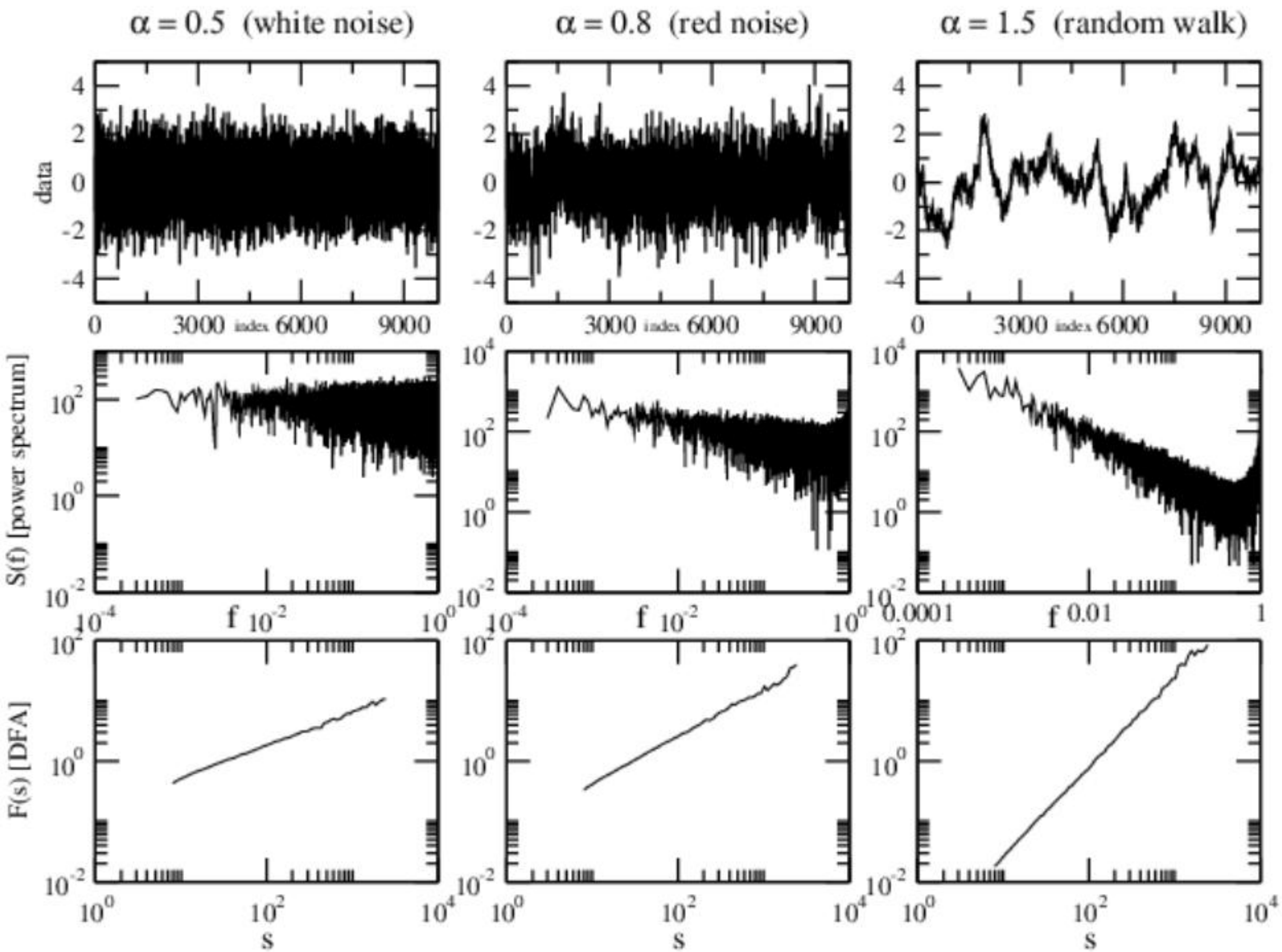
# How to study tipping points: data analysis & modelling

- Recording measurements with time:
  time series

- Analysing time series:
  metrics and quantifiable properties

- Modelling time series:
  stochastic models

# Temporal scaling: power-law correlations



# Power-law correlations

**Real geophysical data (air temperature, SST, river flux, etc.) carry memory caused by various types of inertia. Statistically, the memory is described in terms of correlations, and there exist several methods to estimate them:**

**1) Power spectrum exponent**

$$q_f = \sum_{k=0}^{N-1} u_k \exp(2\pi ikf / N), \;\; S(f) = |q_f|^2 + |q_{N-f}|^2 \propto f^{-\beta}$$

**2) Auto-correlation function (ACF) exponent**

$$C(s) = \langle \varphi_i \varphi_{i+s} \rangle = \frac{1}{N-s} \sum_{i=1}^{N-s} \varphi_i \varphi_{i+s}$$

**3) Detrended fluctuation analysis (DFA) exponent**

$$F(s) = \sqrt{\frac{1}{2N} \sum_{\nu=1}^{2N} \frac{1}{s} \sum_{((\nu-1)s+1}^{\nu s} \left[ Y_\nu(i) - p_\nu^k(i) \right]^2} \propto s^\alpha$$

$$\alpha = 1 - \gamma / 2 = (1+\beta)/2,$$

$\alpha = 0.5$ – uncorrelated data

$\alpha > 0.5$ – correlated data

$\alpha = 1.5$ – random walk with uncorrelated steps

Barabasi and Stanley, *Fractal Concepts in Surface Growth*, Cambridge University Press, 1995

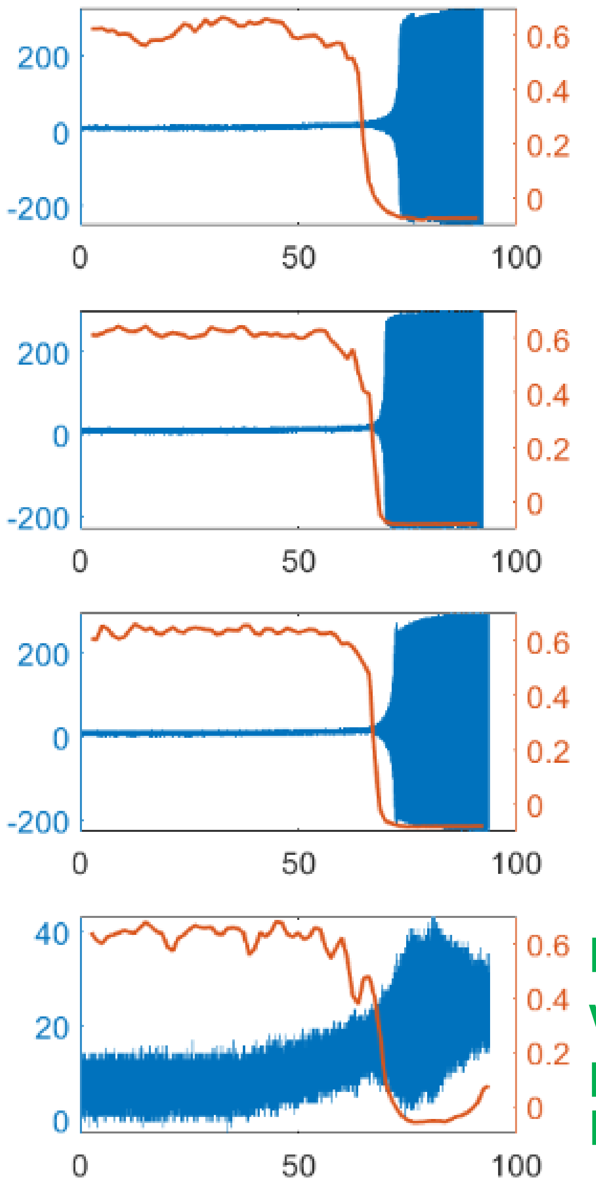
# Relationship between scaling exponents α, β, γ and Hurst exponent H

Madras uni group of Prof. Sujith: thermoacoustic experiment in a Rijke tube

1-α EWS indicator is identical to H-indicator, which illustrates the known connection of H with the three scaling exponents, ***1-α = H***

$$\alpha = 1 - \gamma/2 = (1+\beta)/2$$

(see publications by Barabasi, Stanley, Havin, and others)




**Radhakrishnan et al, Early warnings are too late when parameters change rapidly, Nature Scientific Reports 2025**

# Potential analysis model

$$\dot{z}(t) = -U'(z) + \sigma\eta$$

$$U(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z$$

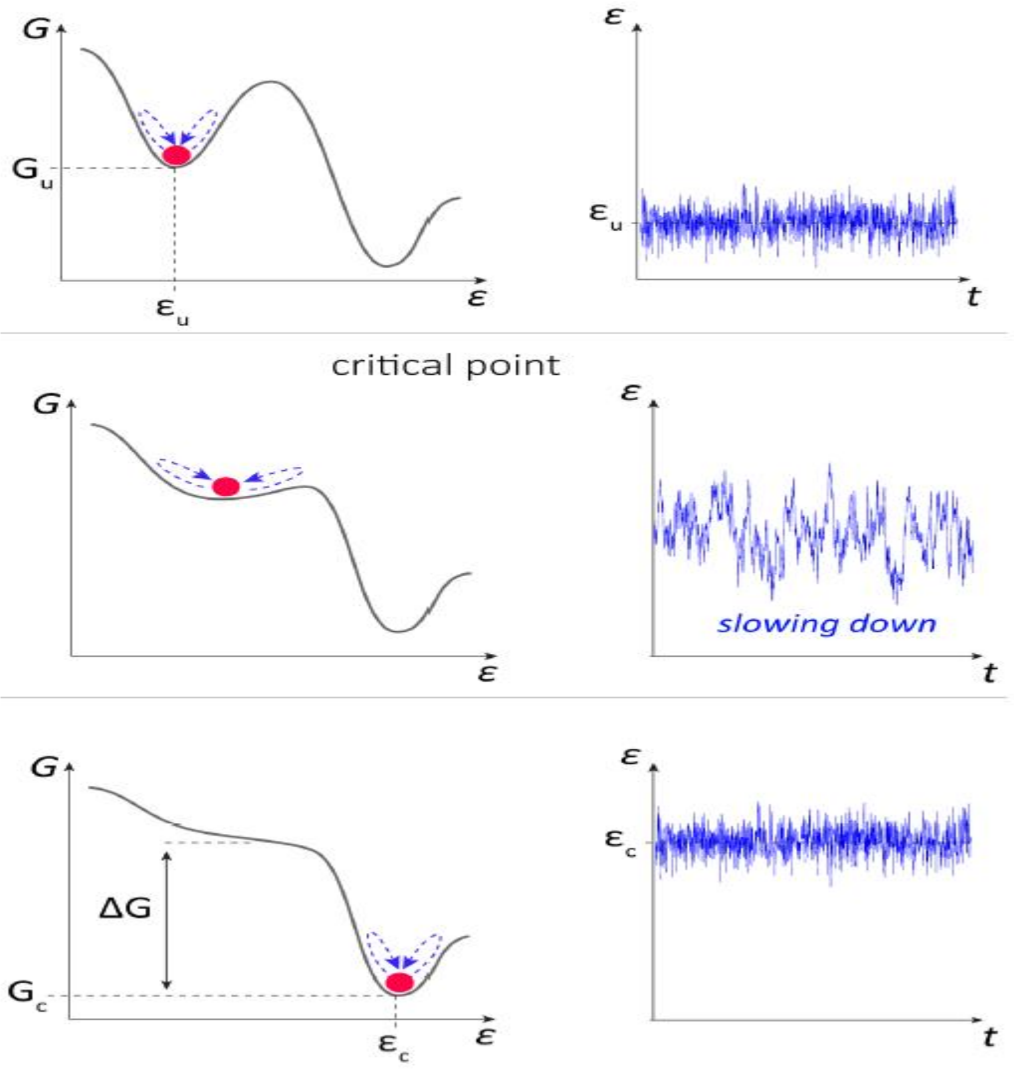
double-well potential in paleoclimate data

Kwasniok & Lohmann, Phys Rev E, 2009

Livina et al, Climate of the Past 2010

# Bifurcating potential

energy
vs
variable




variable
vs
time

# Potential & probability density

Fokker-Planck equation

$$\partial_t p(z,t) = \partial_z [U'(z) p(z,t)] + \frac{1}{2}\sigma^2 \partial_z^2 p(z,t)$$

$$p(z) \approx \exp[-2U(z)/\sigma^2]$$

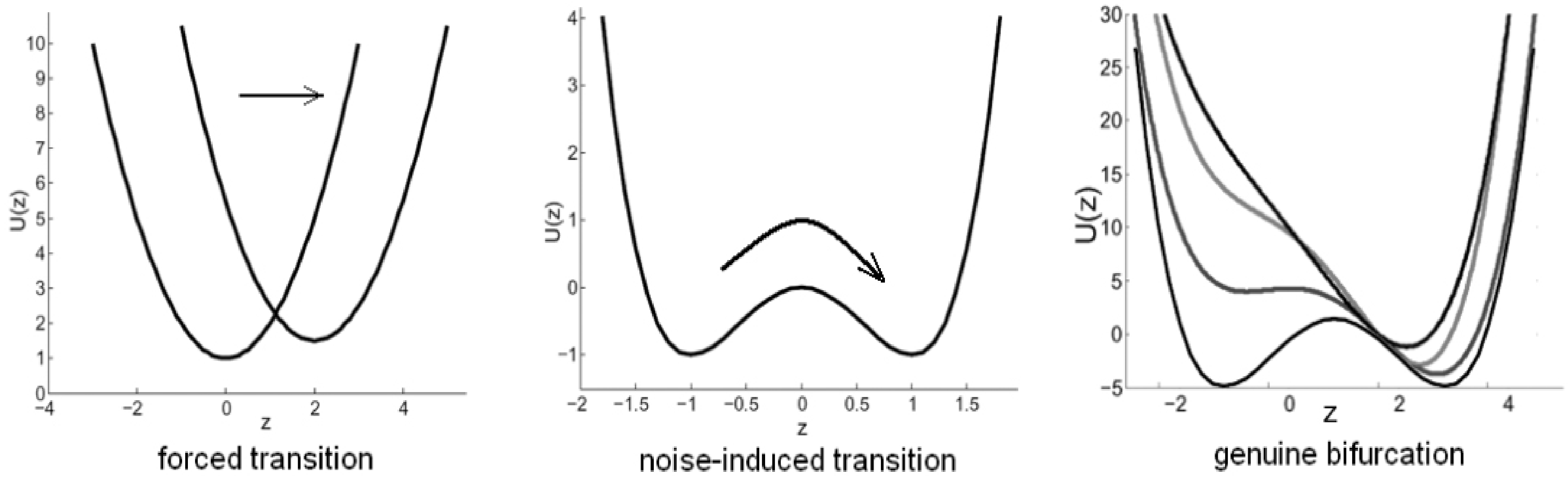
If we assume that the considered subset of data is stationary, then

$$U = -\frac{\sigma^2}{2} \log p_d$$

bimodal histogram $\Leftrightarrow$ double-well potential

# Transitions and bifurcations

**These can be classified in terms of the system potential, which defines the states of a climatic variable**

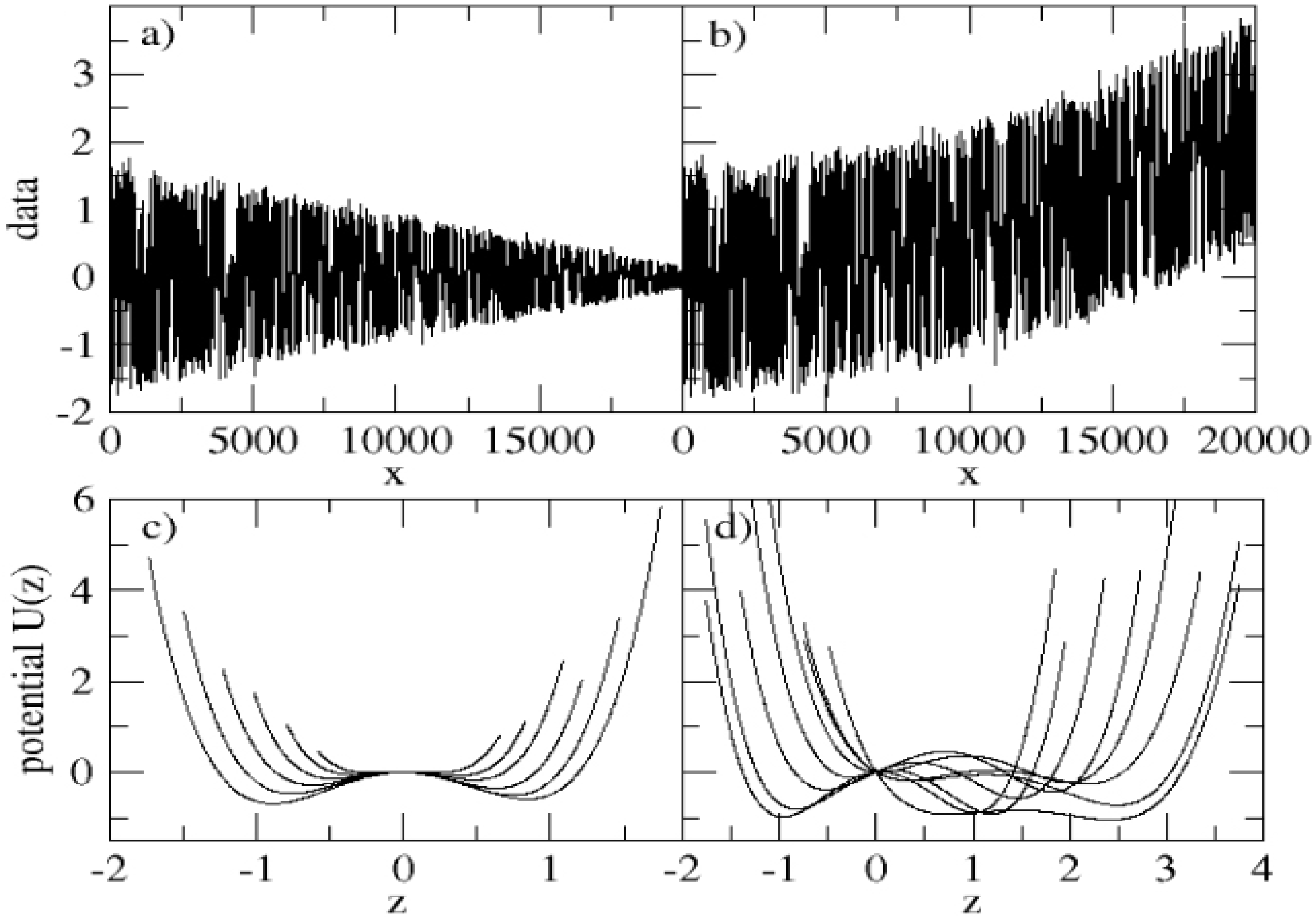



**Tipping elements of the Earth system may approach tipping points that may be transitions as well as bifurcations**

Livina et al, Clim. Dyn. 2011

Ashwin et al, Phil. Trans. Royal Soc A, 2012: r-tipping, b-tipping, n-tipping

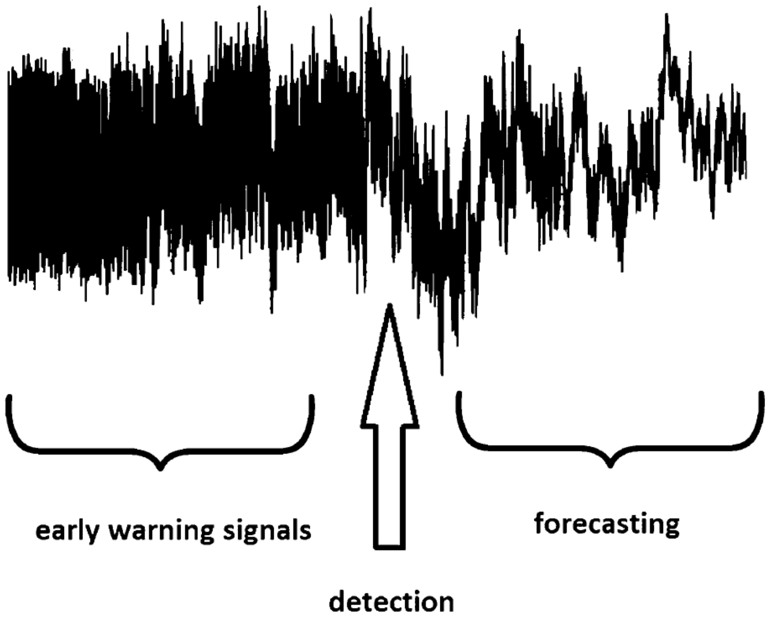
# Examples of artificial data

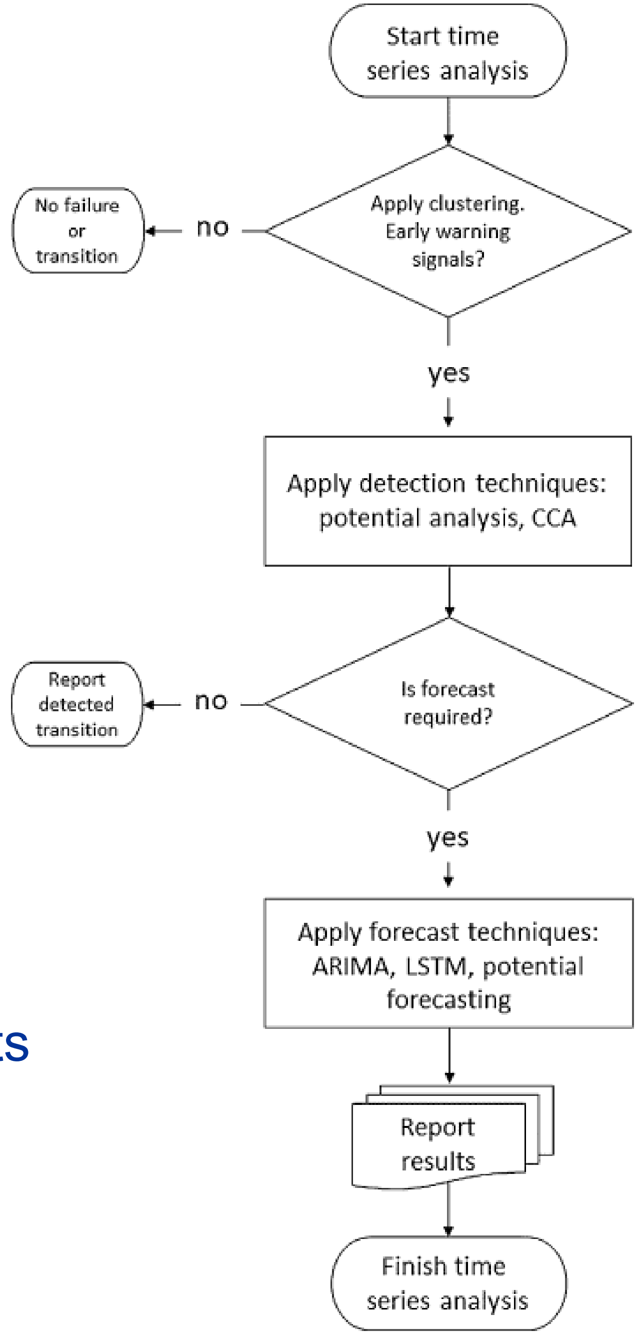



bifurcation

forced transition

# Workflow of analysis




- Anticipating: early warning signals of tipping points
- Detecting: potential analysis
- Forecasting: PDF & potential analysis

Billuroglu & Livina, JFAP 2022

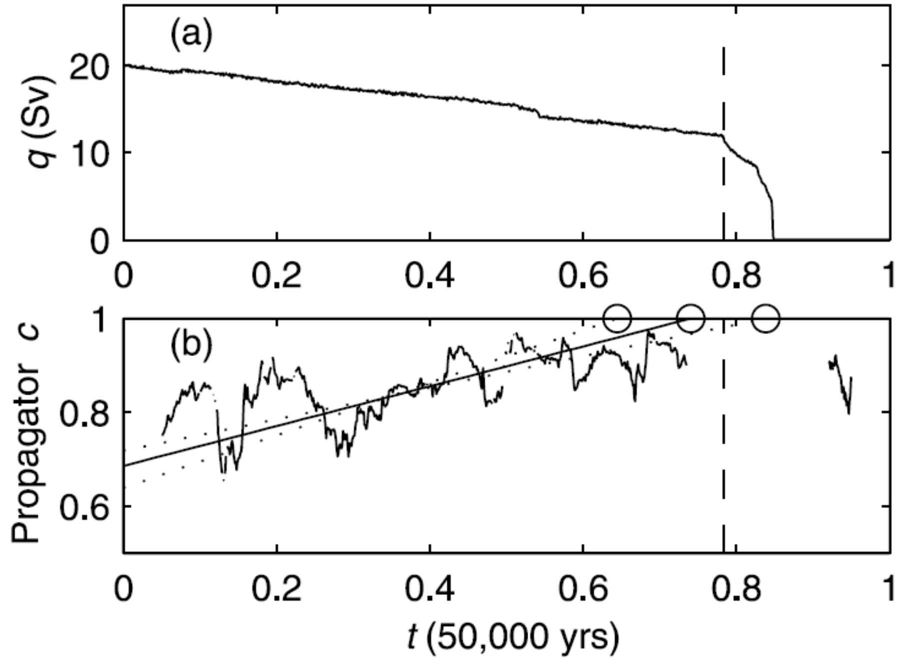
# Anticipating tipping points

# Early warning signals

- ACF-indicator based on lag-1 autocorrelation (Held & Kleinen, GRL 2004)
- DFA-indicator based on detrended fluctuation analysis exponent in short-term range (Livina & Lenton, GRL 2007)
- PS-indicator based on power spectrum (Prettyman et al 2018)
- Variance and skewness (note: variance may drop when tipping point is approaching due to finite-size effects).
- EWS indicator based on hierarchical clustering of subsets of time series (Billuroglu & Livina, 2022)
- Trends in indicators can be assessed using Mann-Kendall rank coefficient

# Degenerate fingerprinting: EWS model

## Held & Kleinen, GRL 2004




aggregation with $\Delta t \approx 1/\kappa$

**North Atlantic stream function from CLIMBER2 model and propagator used to detect bifurcation; THC collapse due to linear increase of $CO_2$ and statistically perturbed increased fresh water forcing**

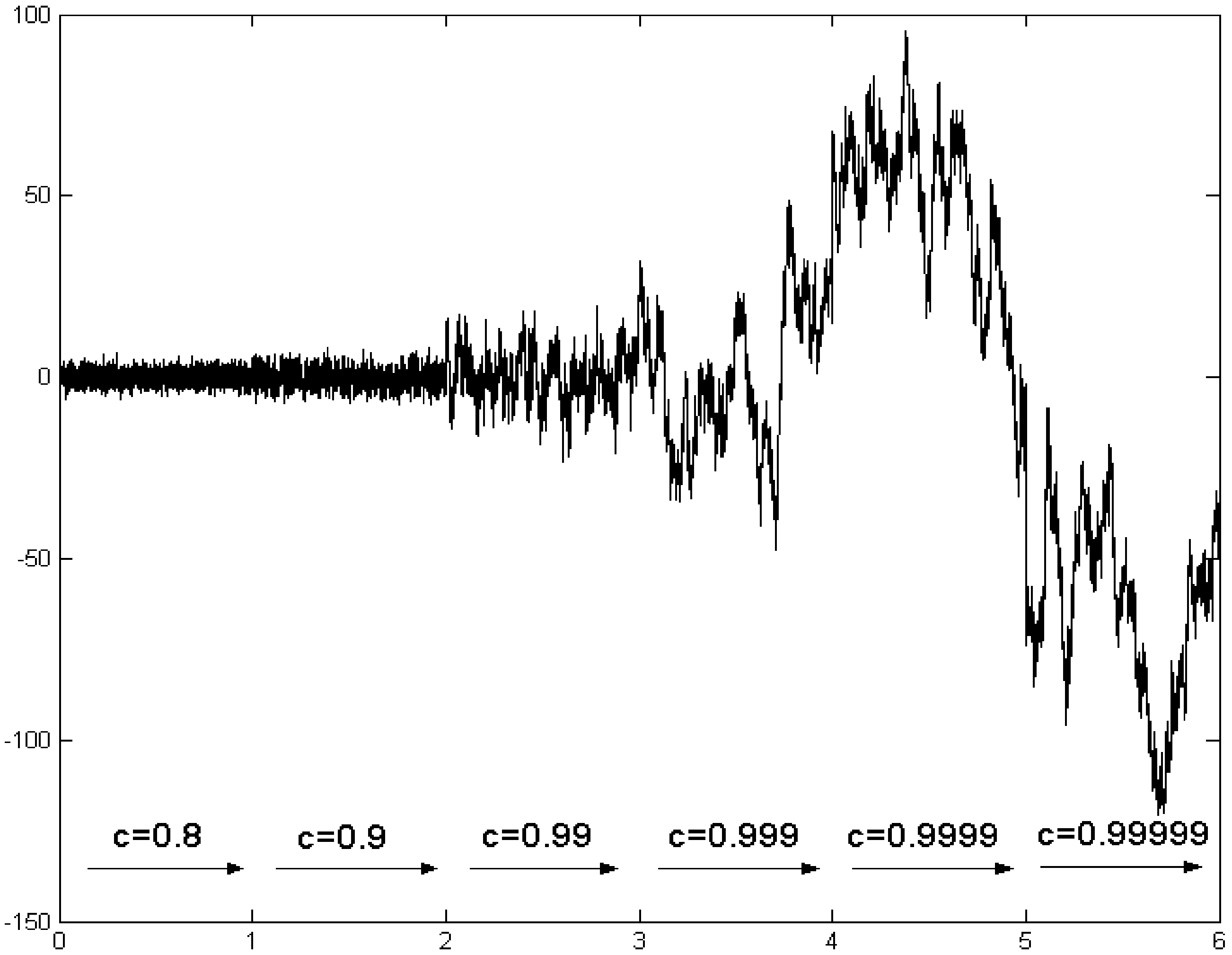
**Series is approximated by an AR(1) process, and exponential decay of the auto-correlation function (ACF) is estimated. Thus ACF-propagator *c* is defined; its gradual trend towards value 1 indicates critical behaviour.**

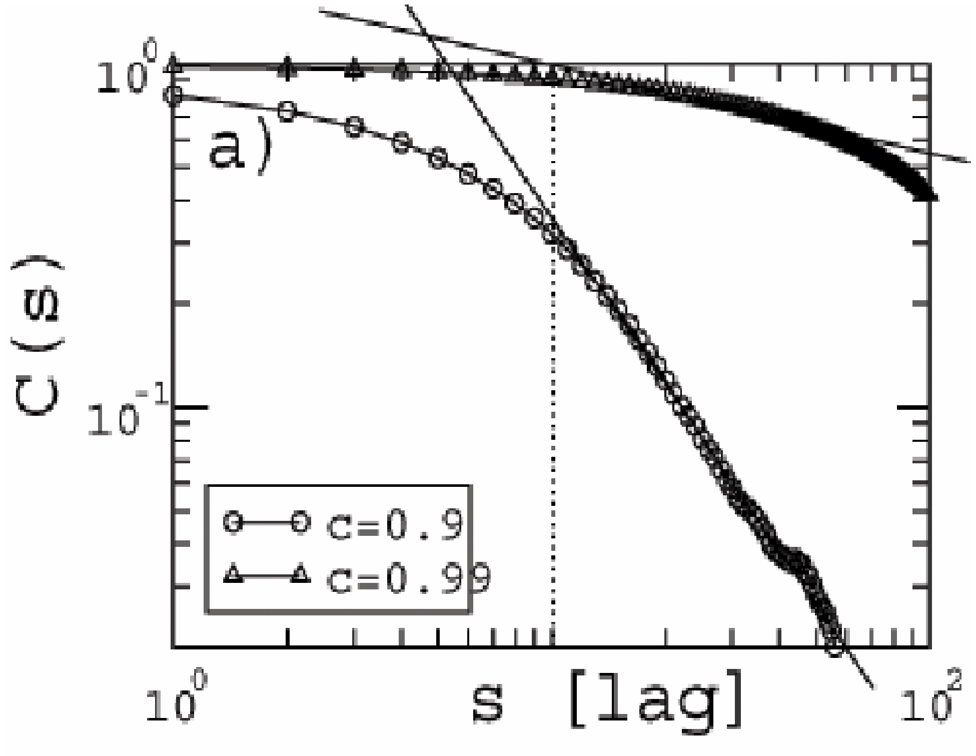
$$y_{n+1} = c y_n + \sigma \eta_n,$$

$$c = \exp(-\kappa \Delta t),\ \kappa \text{ is decay rate}$$

$$(\kappa = 0 \text{ when } c = 1)$$

# AR(1) data at critical values *c*

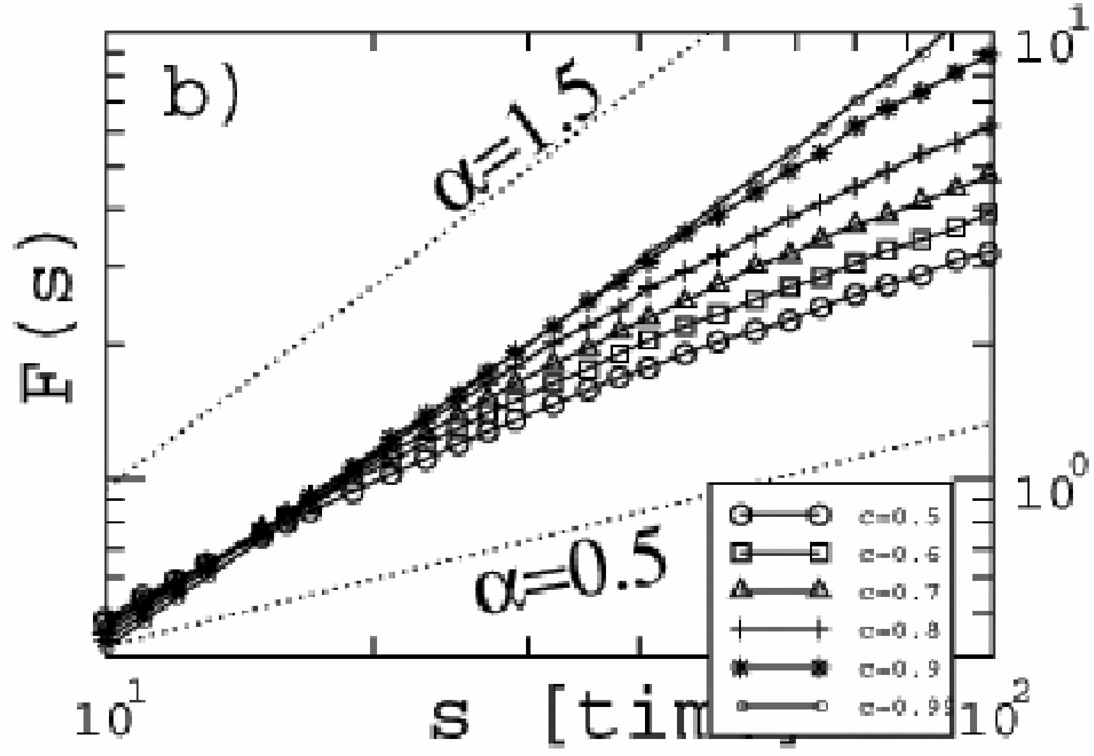



Livina & Lenton, GRL 2007

**Increasing c – increasing nonstationarities – increasing short-term memory**

# DFA of AR(1) data at various *c*

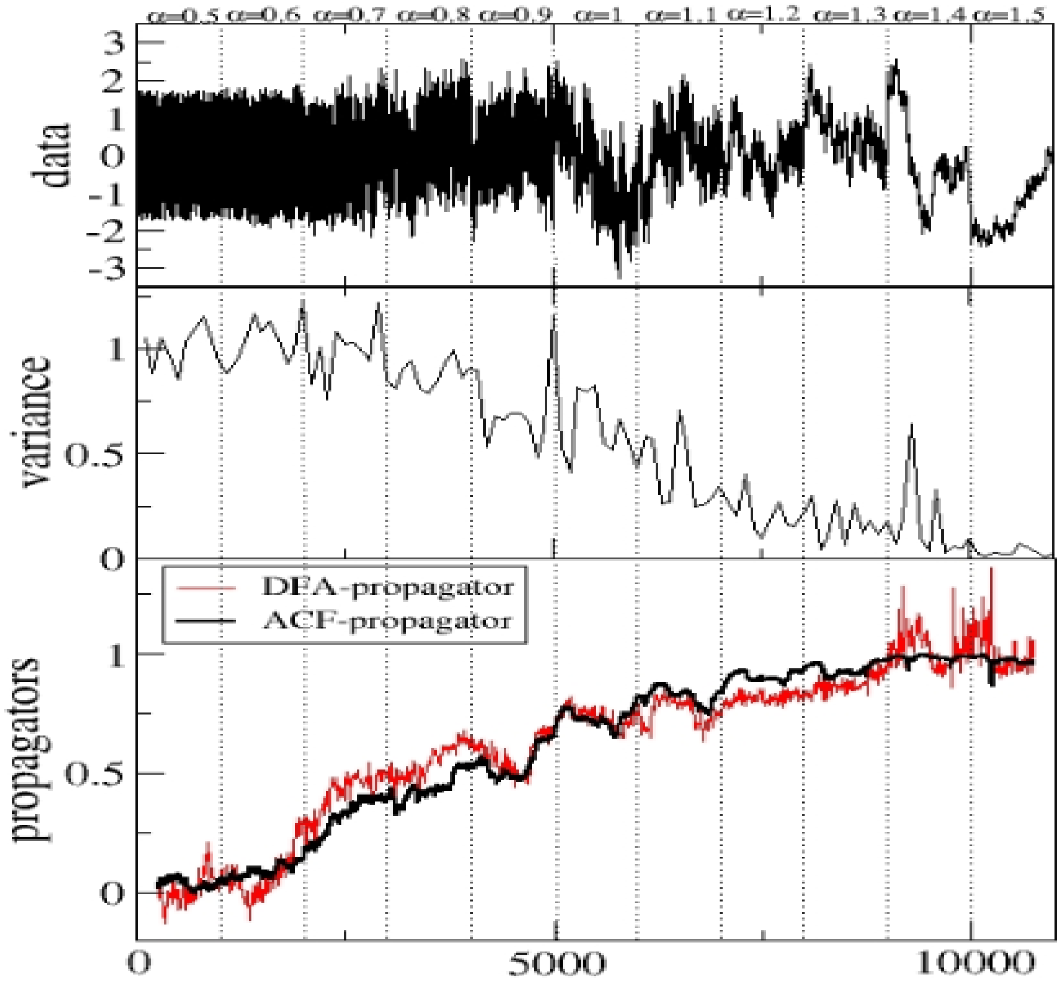



**AR(1) process** **is not** **long-term correlated ( Bogachev et al, 2009)**

**The short-memory effects are observed for 10-100 time units**

# AD with increasing memory

Livina, Ditlevsen, Lenton (Physica A, 2012)

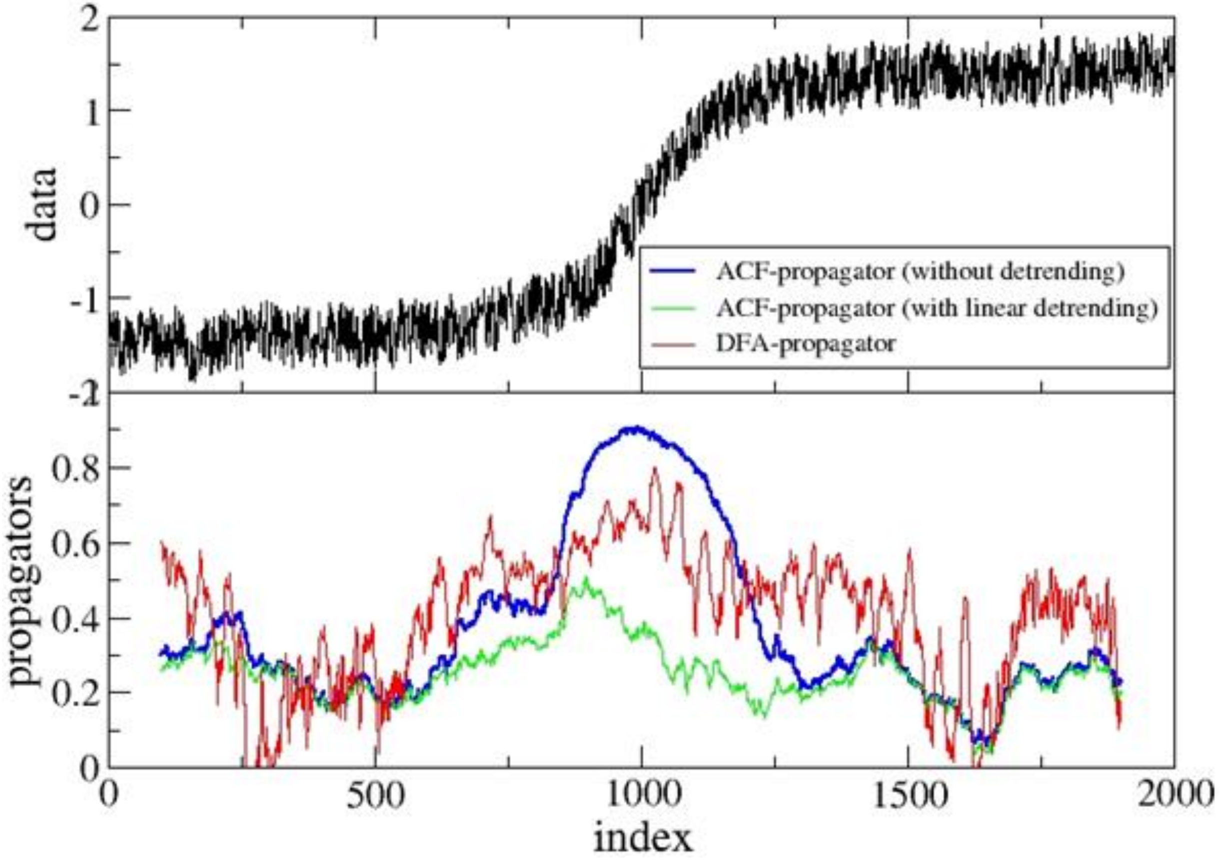



**When ACF-indicator reaches critical value 1, DFA-indicator is still capable to reflect the variability in the variance**

# Example of transition: sigmoid

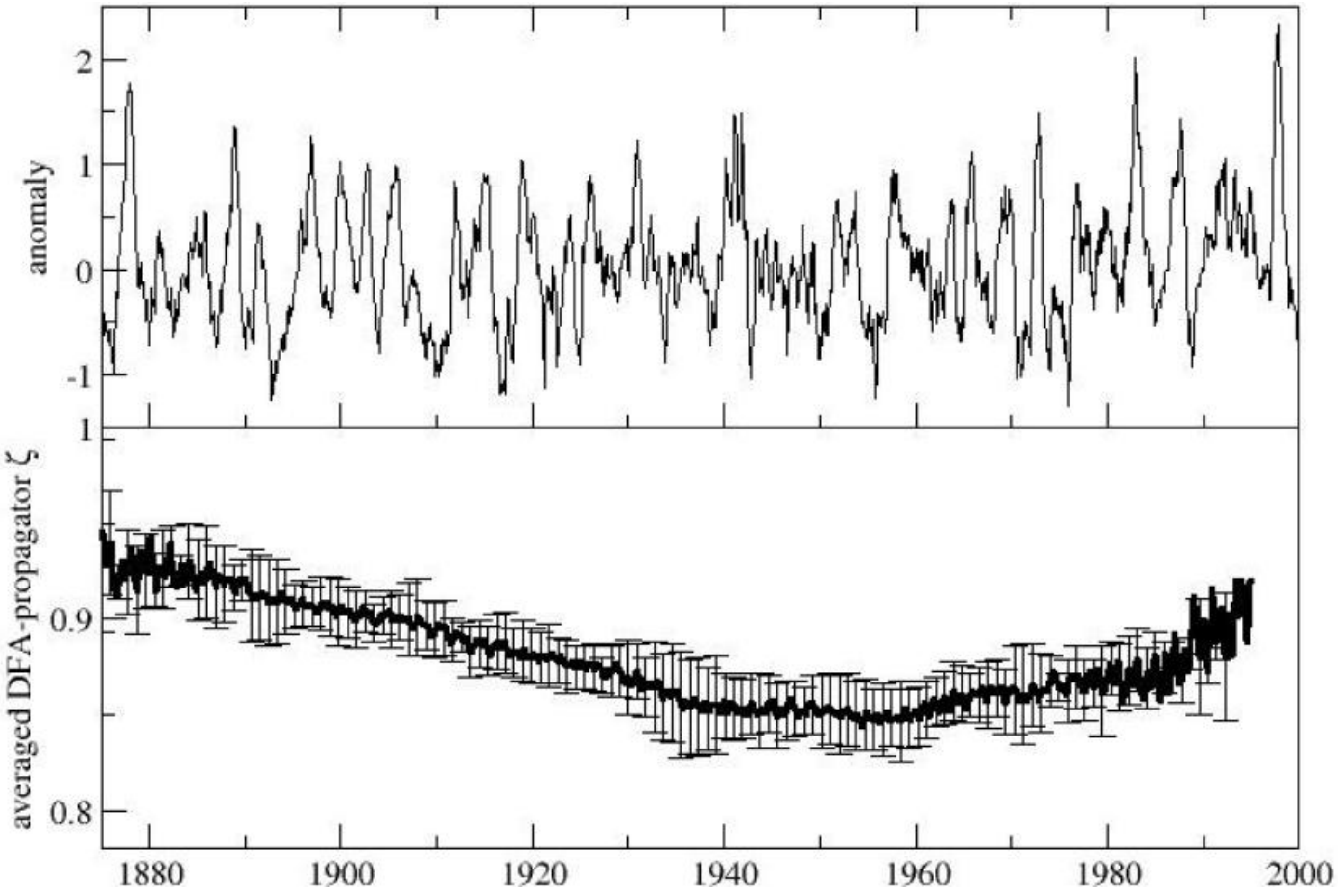
added red noise, fluctuation exponent 0.7

Livina, Ditlevsen, Lenton (Physica A, 2012)




ACF-propagator without detrending is sensitive to transitions

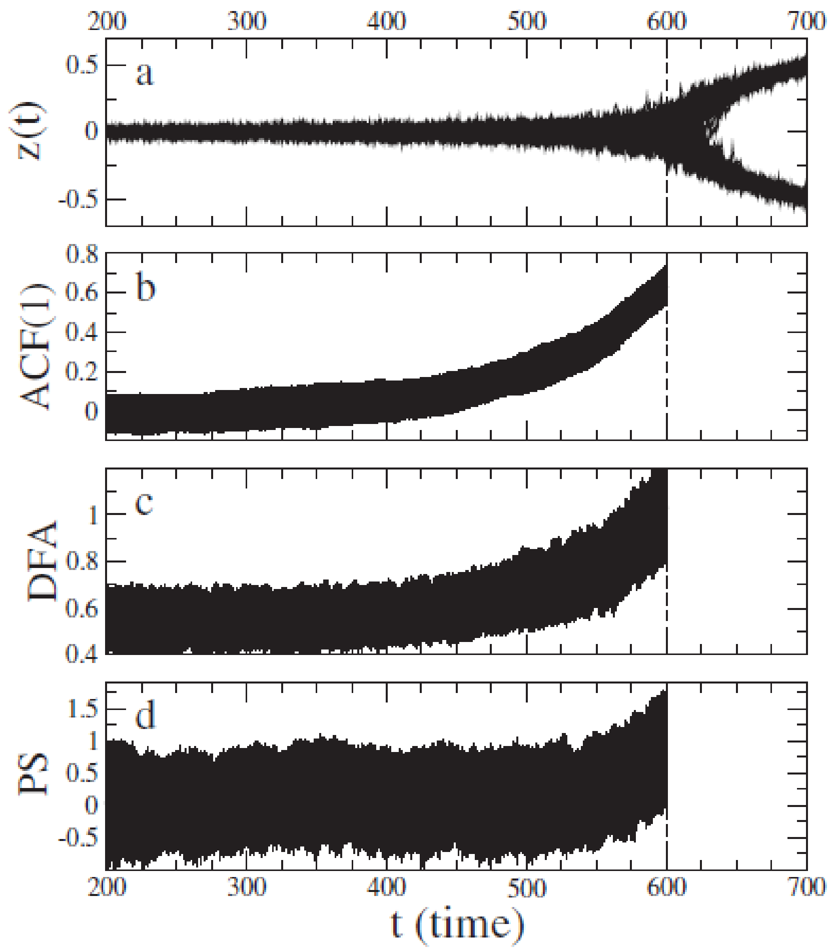
# Uncertainty quantification: variable window size



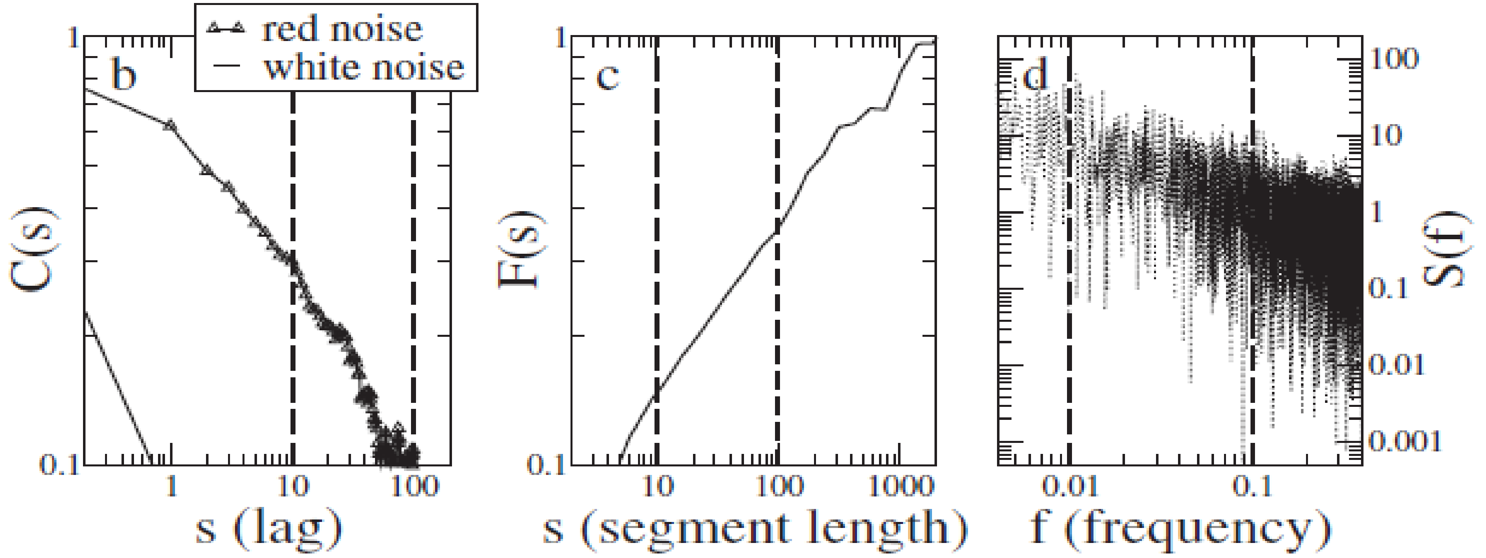

SST anomaly, monthly data

**Window size vary between 5% and 50% of the data length**

# Power-spectrum EWS indicator



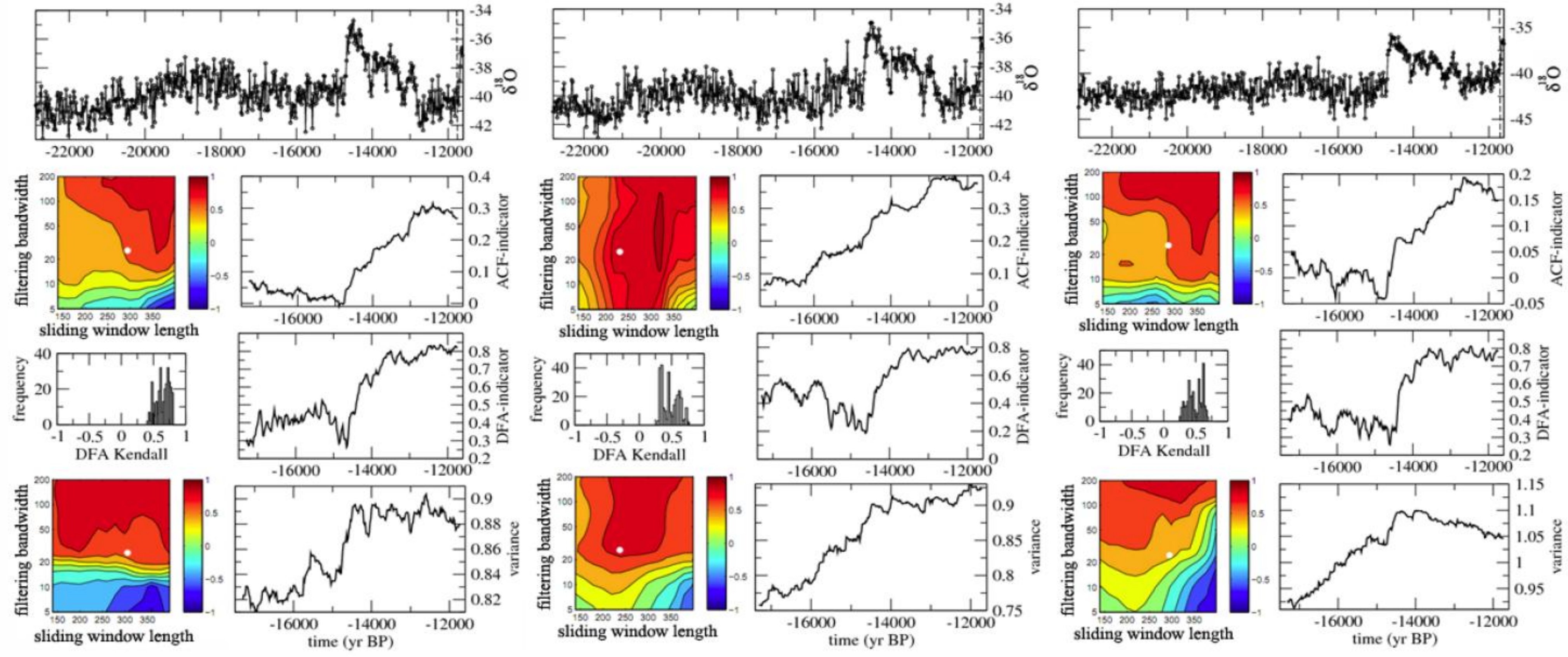

$$\dot{z}(t) = -\frac{\partial}{\partial z}\left(z^4 + \left(3 - \frac{t}{200}\right) z^2\right) + \sigma \eta_t$$




Prettyman et al, EPL 2018
Prettyman et al, Chaos 2019
Prettyman et al, ERL 2021

# Sensitivity analysis: Greenland data

GRIP GISP2 NGRIP



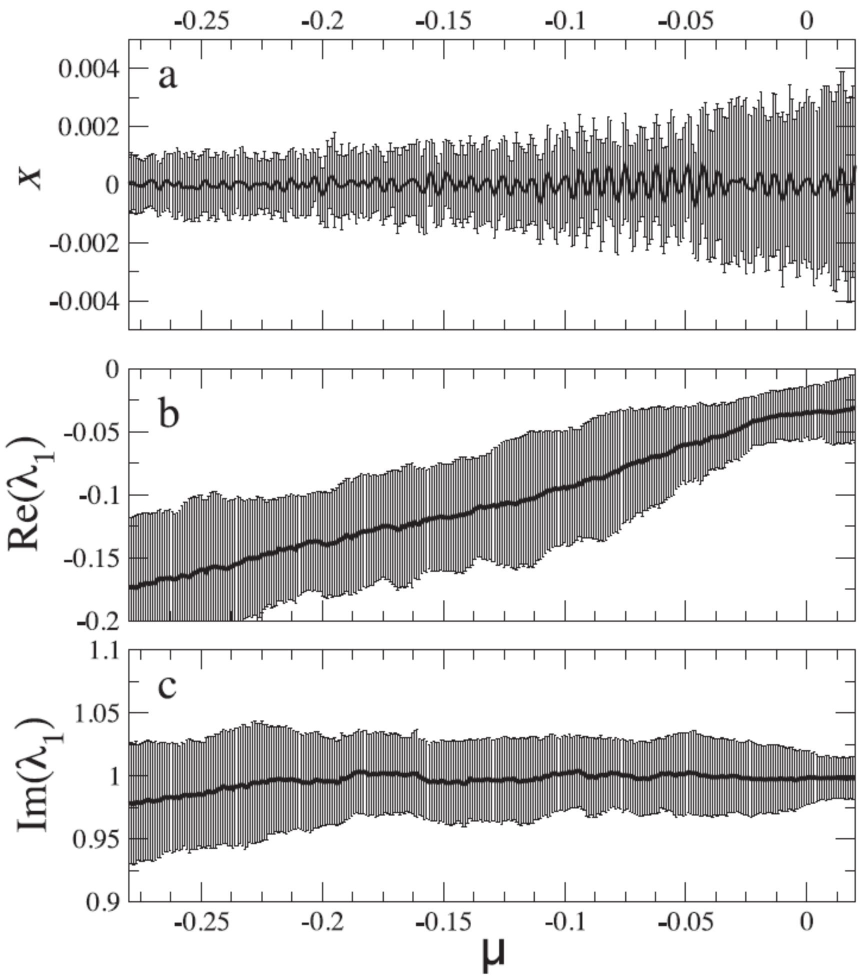

Lenton et al, CoP 2012

# Multivariate EWS

- Principal Component Analysis can be applied to data fields (the same variable over several dimensions).

- Higher-dimensional analogue of ACF-indicator can be applied using eigenvalues of the system Jacobian

Williamson & Lenton, Chaos 2015
Prettyman et al, Chaos 2019




40 trajectories of the van der Pol oscillator with EWS trend in the real part of the first eigenvalue of the system Jacobian

# EWS indicator may inform of different processes

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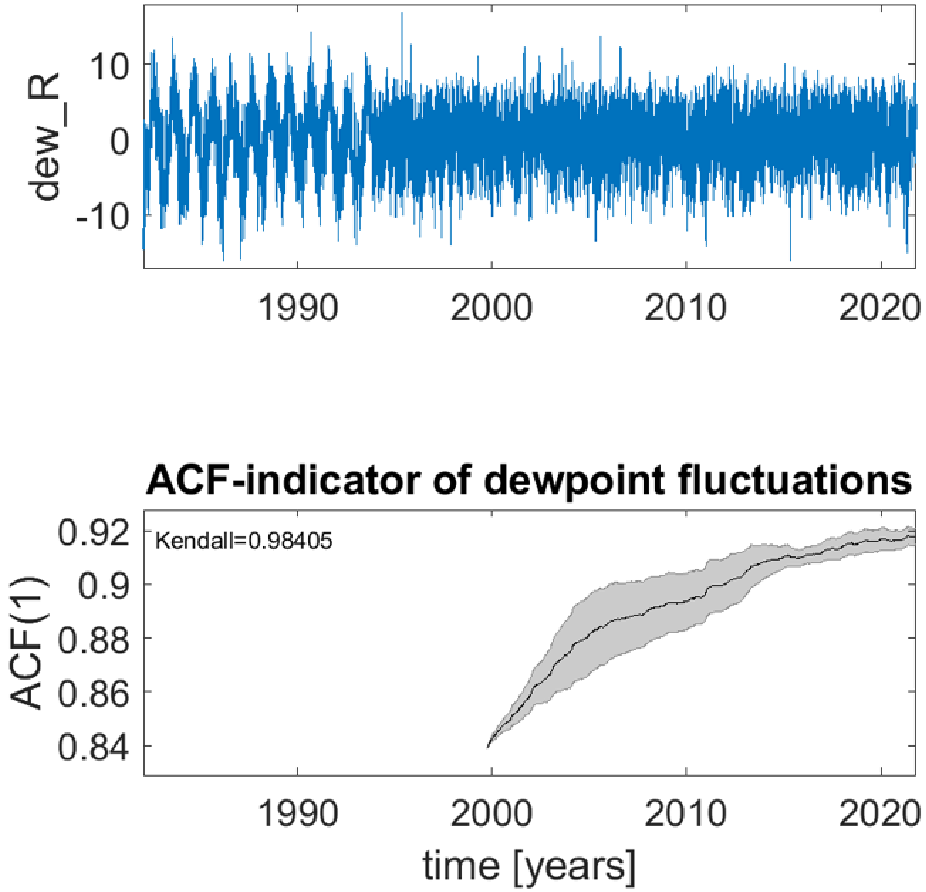
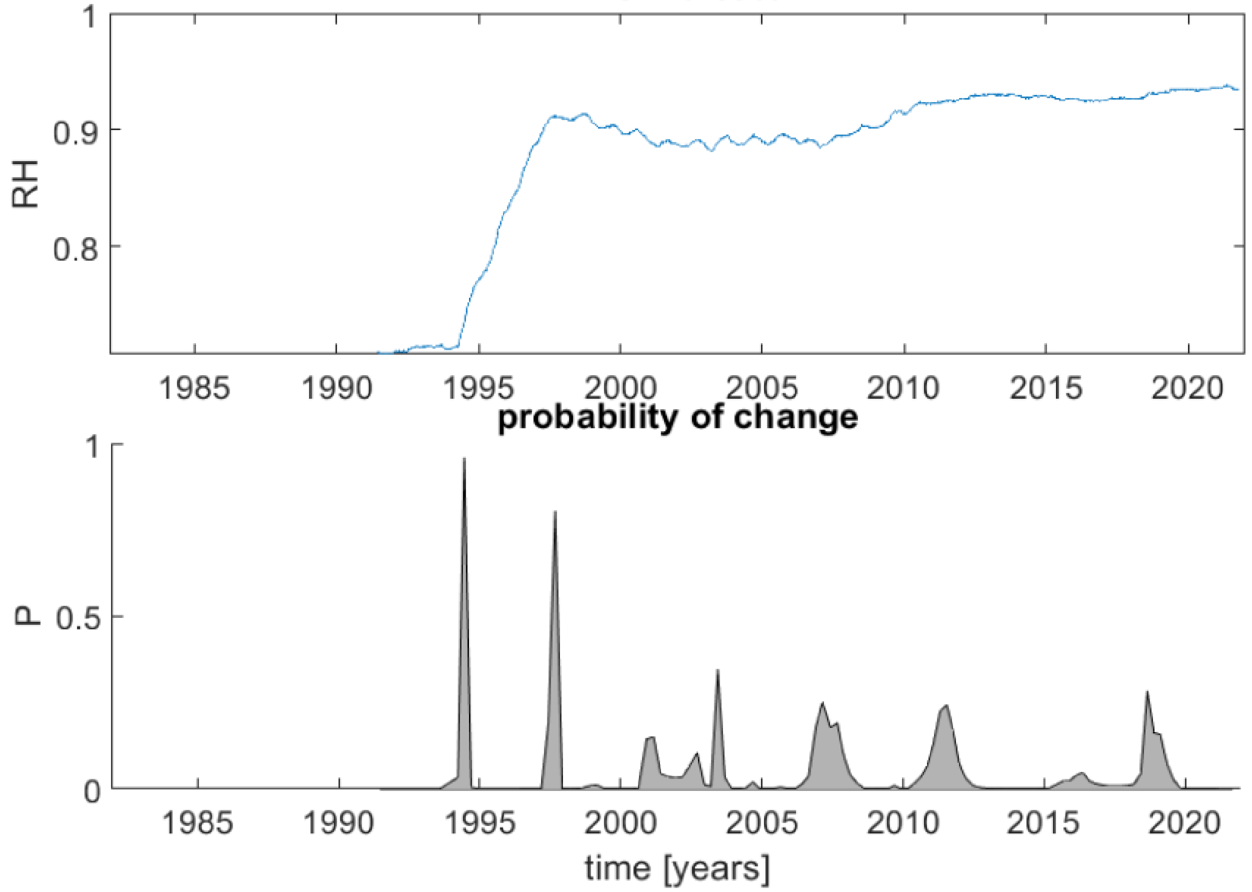



Livina et al, Tipping point analysis helps identify sensor phenomena in humidity data, Geoscientific Instrumentation, Methods and Data Systems, 2025

Supplemented EWS with probabilistic method based upon (Zhao et al, RSoE, 2019) - Bayesian Change Point detection method applied to indicator detected sensor changes

# Preprocessing data in scaling analysis

NPL

- Held & Kleinen used aggregation of data to reduce the effect of weather noise
- Dakos et al used residuals after applying Gaussian filter
- Alternatively, it is possible to use wavelet denoising for the same.
- When there are gaps or poor temporal resolution, we cannot interpolate data, because that would introduce spurious correlations in the data, which would affect estimation of lag-1 autocorrelations
- Many datasets are studied in “raw” format

# Detecting tipping points

# Potential & probability density

Fokker-Planck equation

$$\partial_t p(z,t) = \partial_z [U'(z) p(z,t)] + \frac{1}{2}\sigma^2 \partial_z^2 p(z,t)$$

$$p(z) \approx \exp[-2U(z)/\sigma^2]$$

If we assume that the considered subset of data is stationary, then

$$U = -\frac{\sigma^2}{2} \log p_d$$

bimodal histogram $\Longleftrightarrow$ double-well potential

# Potential analysis stages

$$\dot{z}(t) = -U'(z) + \sigma\eta$$

$$U(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z$$

(1) Estimation of the number of states – polynomial degree of U(z)

(2) Estimation of noise level $\sigma$

(3) Derivation of potential coefficients using Unscented Kalman Filter (UKF)

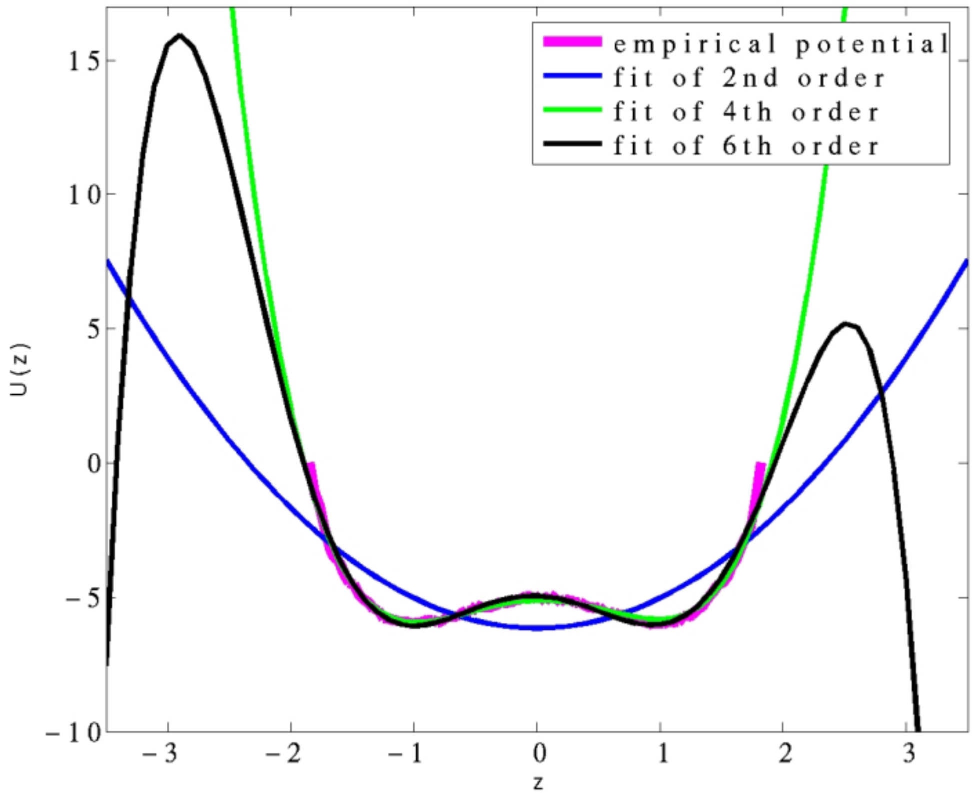
**(1) is the basis of the potential contour plot**
**(2)-(3) are needed for potential curves**

Livina *et al*, *Climate of the Past* 6, 77-82, 2010

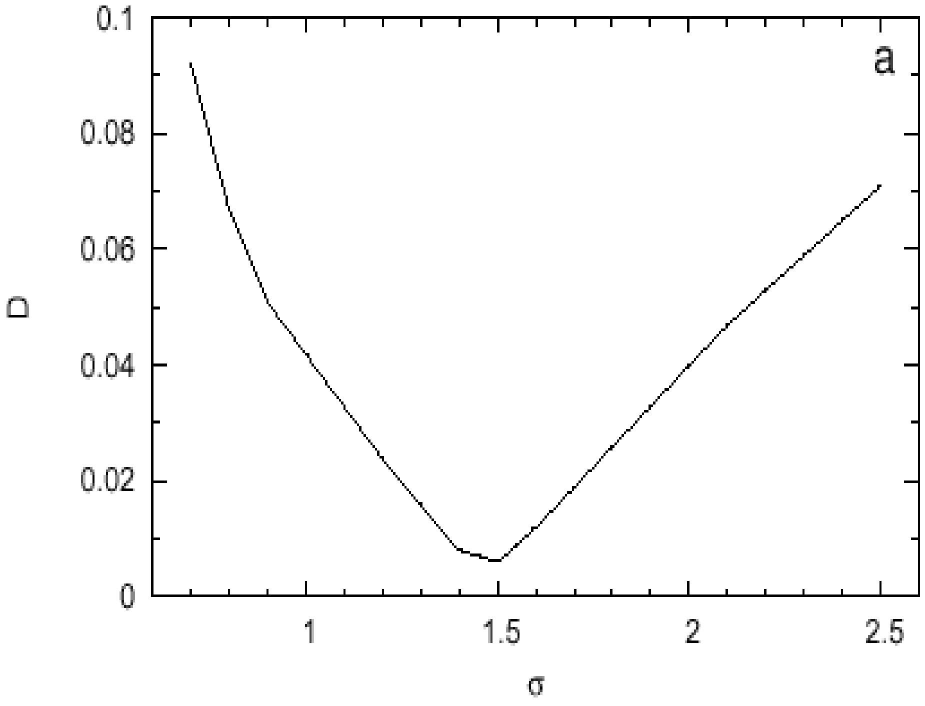
# Estimation of the number of states

Polynomials of increasing even degree are fitted, and the highest degree with a positive leading coefficient is adopted

Livina et al, CoP 2010



# Estimation of the noise level




The UKF procedure is run for different noise levels and the deviation between the cumulative distribution function of the model and the data is monitored as

$$D = \max_{z} \left| \Phi_m(z) - \Phi_d(z) \right|$$

Searching for minimum of this function with respect to noise level provides the final value of noise.

Kwasniok & Lohmann, Phys Rev E 80, 066104, 2009

# Potential coefficients: UKF

Discretised equation with step h $z_t = z_{t-1} - hU'(z_{t-1}) + \sqrt{h}\sigma\eta_t$

Observation equation $y_t = z_t + \varepsilon_t$

Augmented state vector with state variable and parameters $x = (z, a_1, \ldots, a_L)$

Covariance matrix at time t-1 having processed the time series up to time t-1 $P^{xx}_{t-1|t-1}$

Kalman update equations

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + K_t(y_t - \hat{y}_{t|t-1})$$

$$P^{xx}_{t|t} = P^{xx}_{t|t-1} + K_t P^{yy}_{t|t-1} K_t^T$$

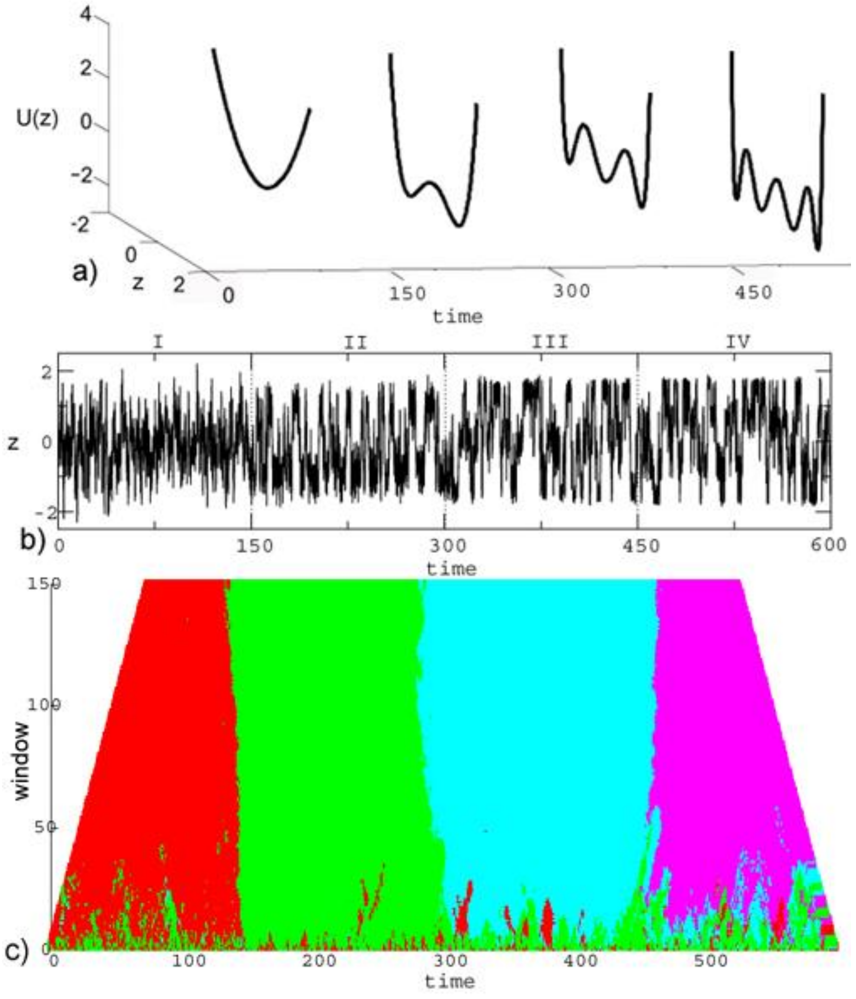
Kalman gain matrix

$$K_t = P^{xy}_{t|t-1}\left(P^{yy}_{t|t-1}\right)^{-1}$$

Chosen sigma points are propagated through the augmented dynamical equation
Leading to transformer means and covarinaces, then updated, until convergence

**Julier et al, IEEE Transactions on automatic control, 2000**
**Kwasniok and Lohmann, Phys Rev E, 2009**

# AD with four potentials




**Potential contour plot at different time scales**

We generate artificial data using Euler scheme

$$x_{t+\Delta t} \approx x_t - \left.\frac{dU}{dx}\right|_t \cdot \Delta t + (W_{t+\Delta t} - W_t)$$

W is a Wiener process

Potentials:

$$U(z) = z^2$$

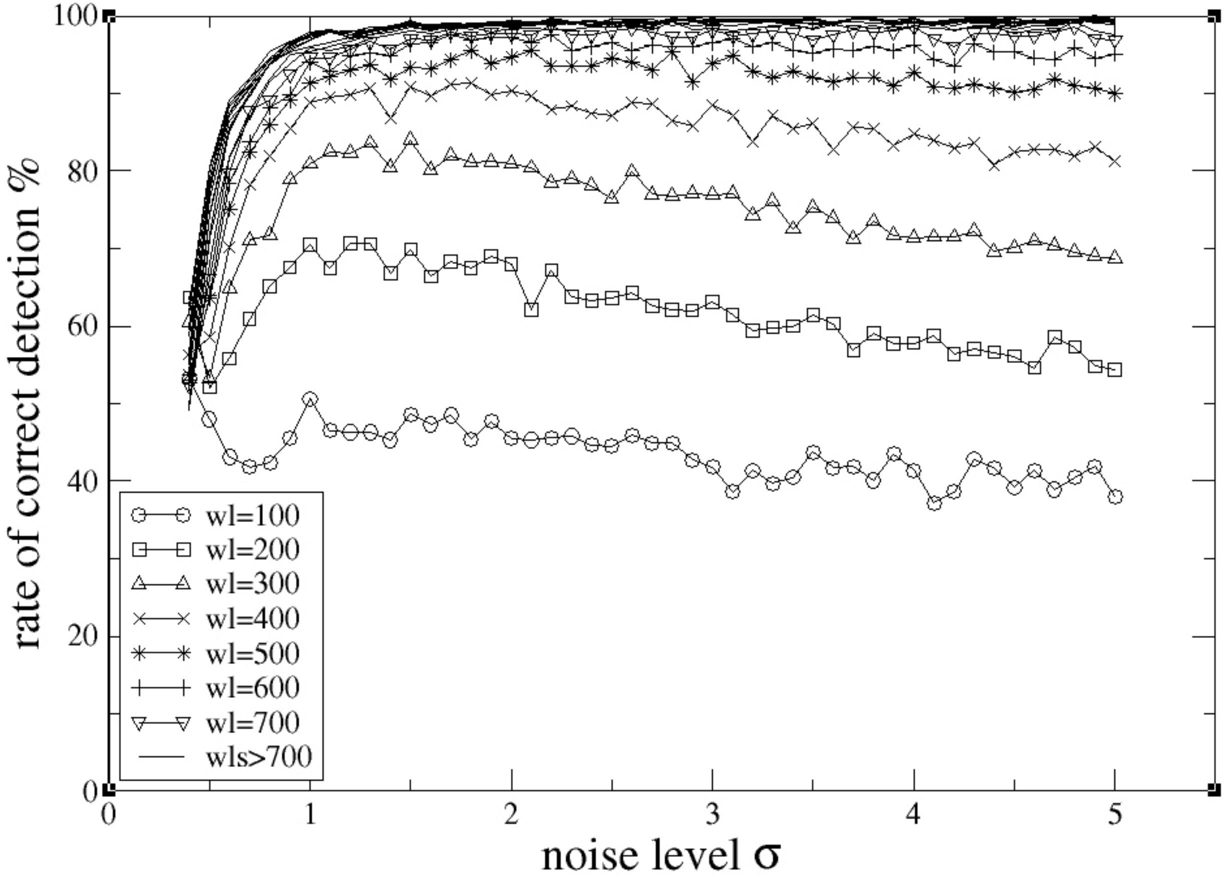
$$U(z) = z^4 - 2z^2$$

$$U(z) = z^6 - 4.5z^4 + 5z^2$$

$$U(z) = z^8 - 6.5z^6 + 13z^4 - 8z^2$$

Livina et al, CoP 2010

# Rate of correct detection of the number of system states

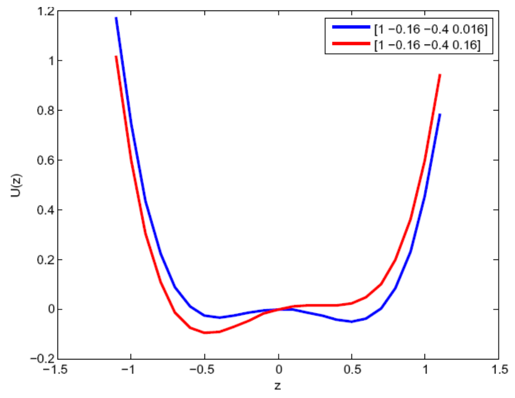



**Detection of two wells in artificial double-well potential data (depth of wells = 1, consider 1000 samples per each value of noise level)**

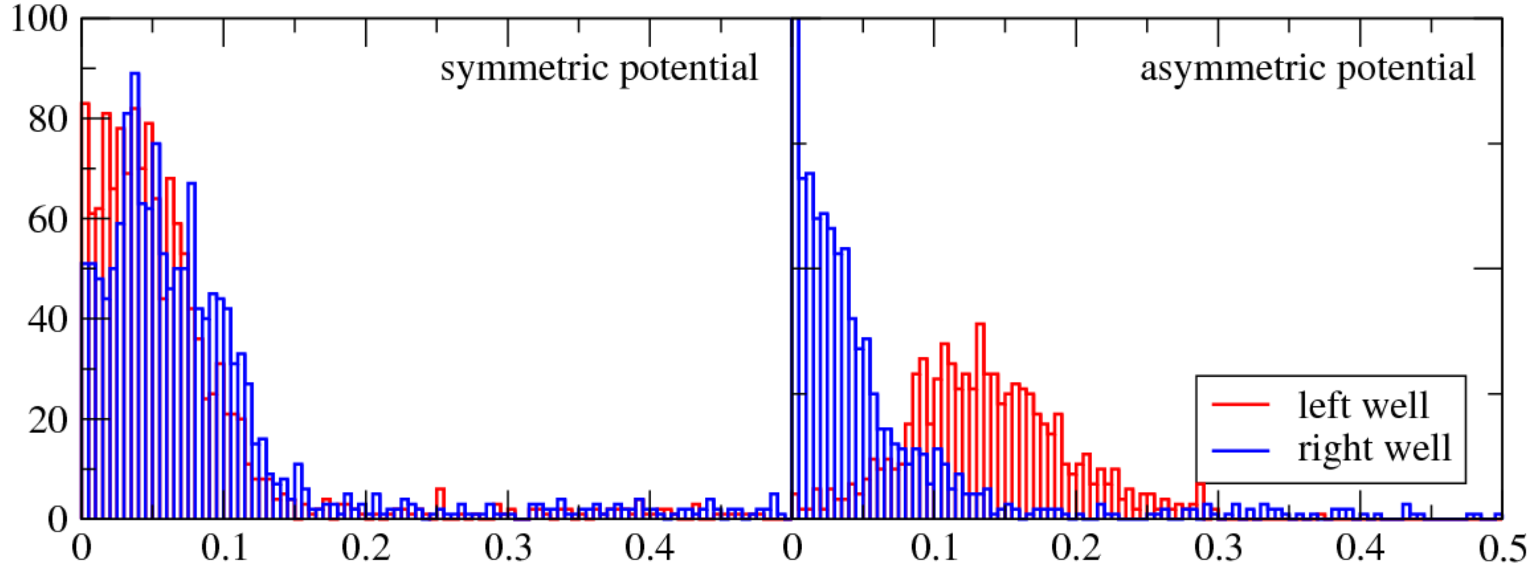
Livina et al, *Climate Dynamics,* 2011

# Artificial data with symmetric and asymmetric potential: comparing detected depths of wells

**sets of 1000K points, wl=5K, σ=0.4**




**Two samples of artificial data and in sliding windows estimated the potentials coefficients, from which we derived the depths of the potential wells. Analytically calculated depths were: for symmetric potential 0.0339 and 0.0488, for asymmetric potential 0.1124 and 0.0007**



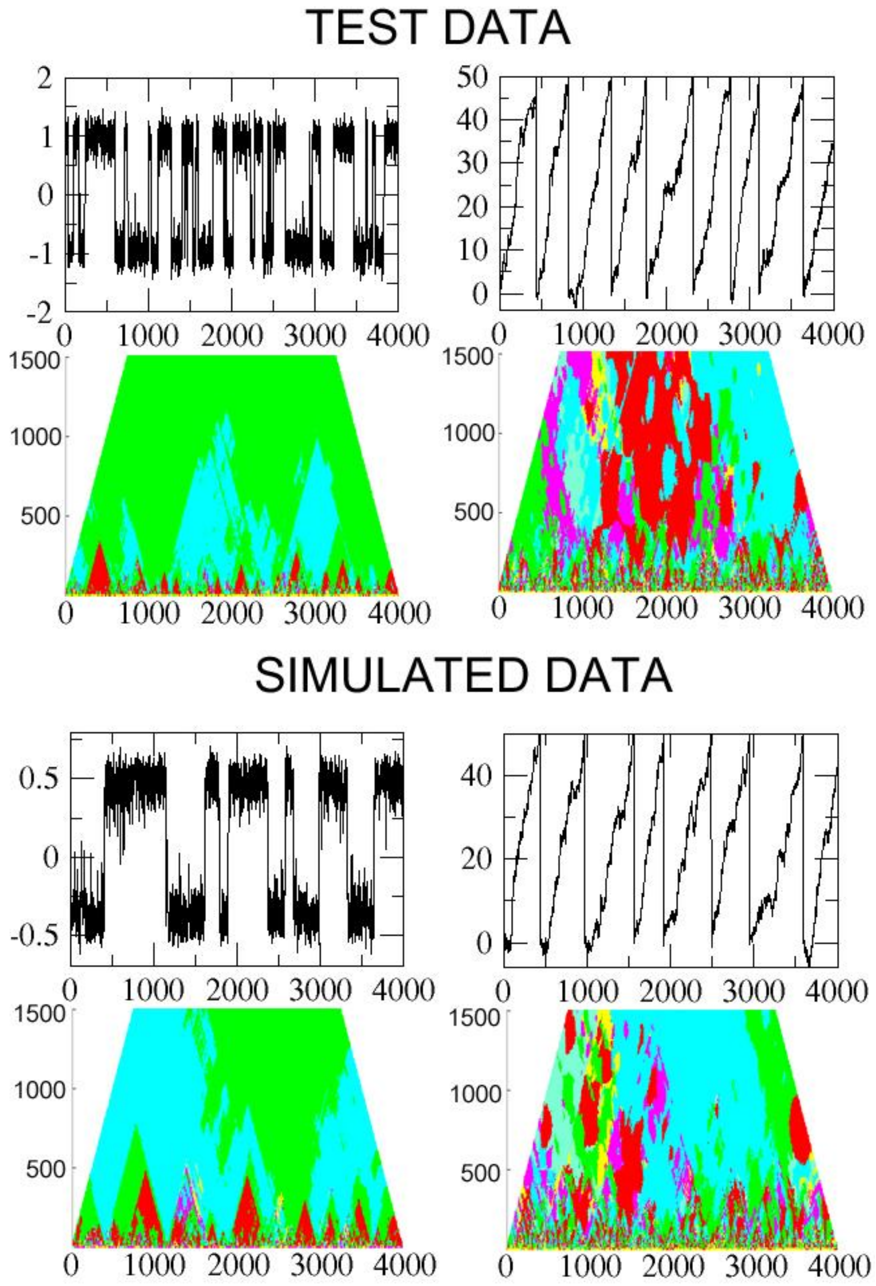
# Blind test with unknown data

Livina, Ditlevsen, Lenton (Physica A, 2012)

- 9 samples of data generated from different (unknown) models provided

- Potential analysis used to try and deduce underlying models, then simulate data equivalent to the test samples

- Method correctly reconstructs generating equation where there is potential behaviour, and recognises sample with non-potential behaviour

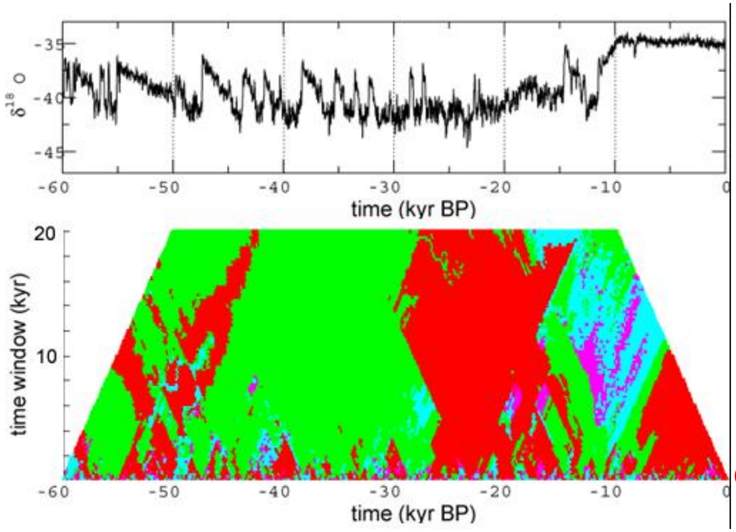



Non-potential system:
a global potential doesn't exist.

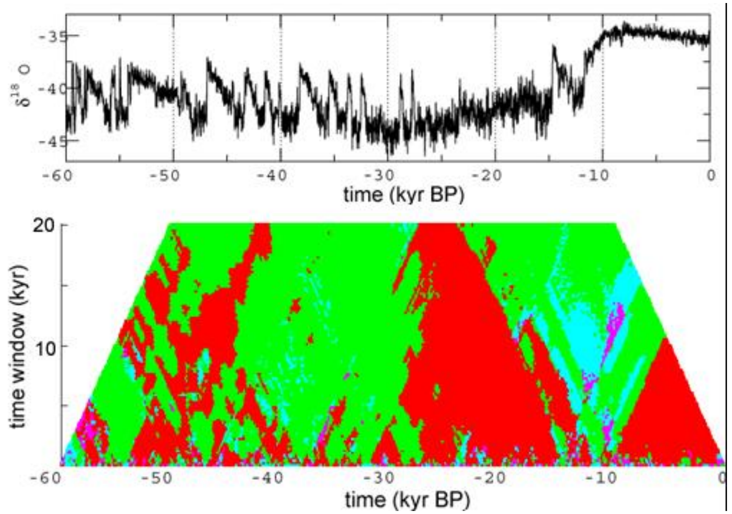
# GRIP & NGRIP temperature proxies

δ[18]O data: bifurcation at 25-28 kyr BP

GRIP



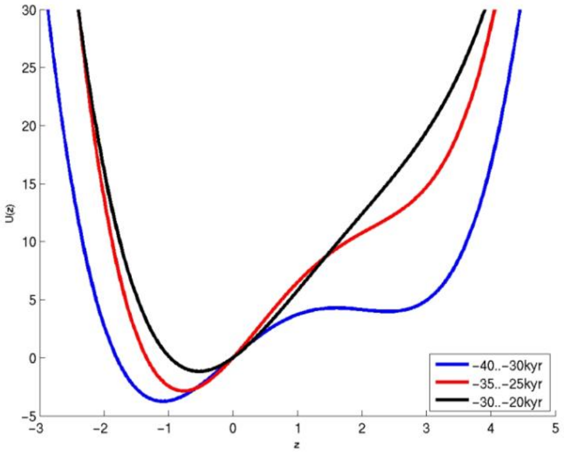

NGRIP




GICC05 time scale, resolution 20yr

(Livina et al, Climate of the past, 2010)

GRIP

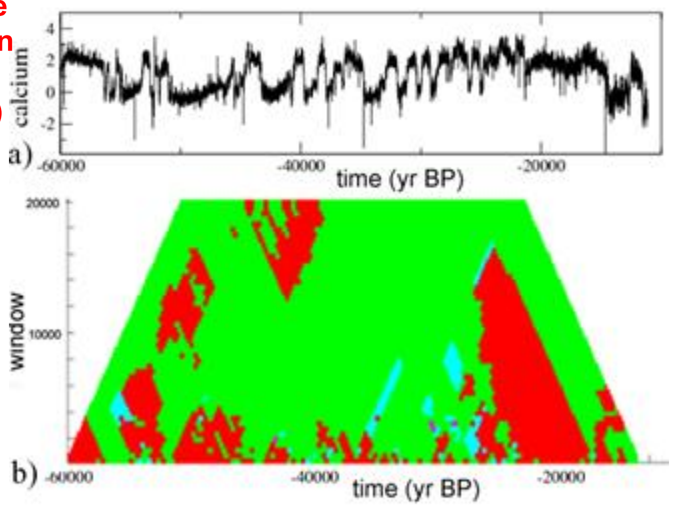



The bifurcation independently confirmed by Cimatoribus et al

(phase space reconstruction In 4D polar coordinates)

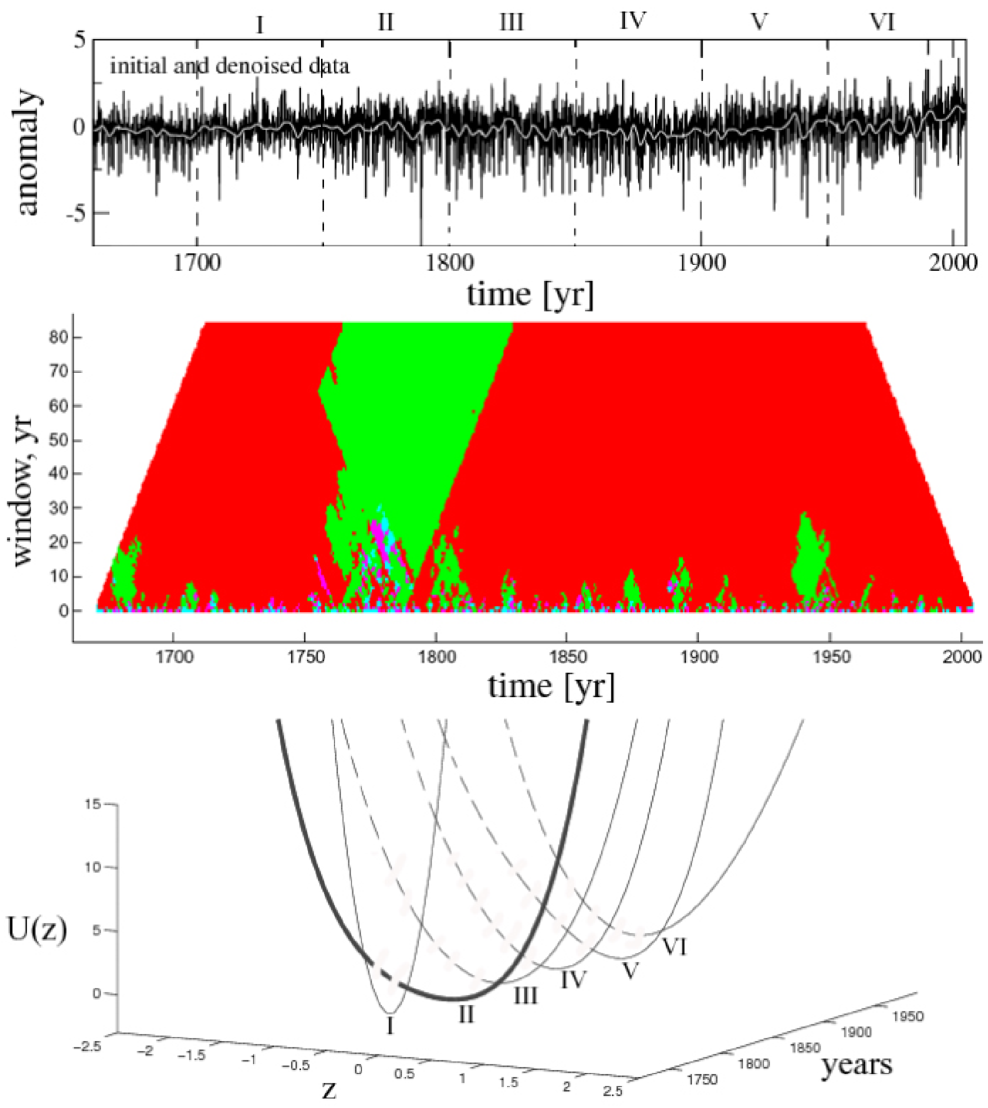
Calcium data: bifurcation at 27-28 kyr BP

GRIP




Annual resolution

# European temperature anomaly 1659-2004

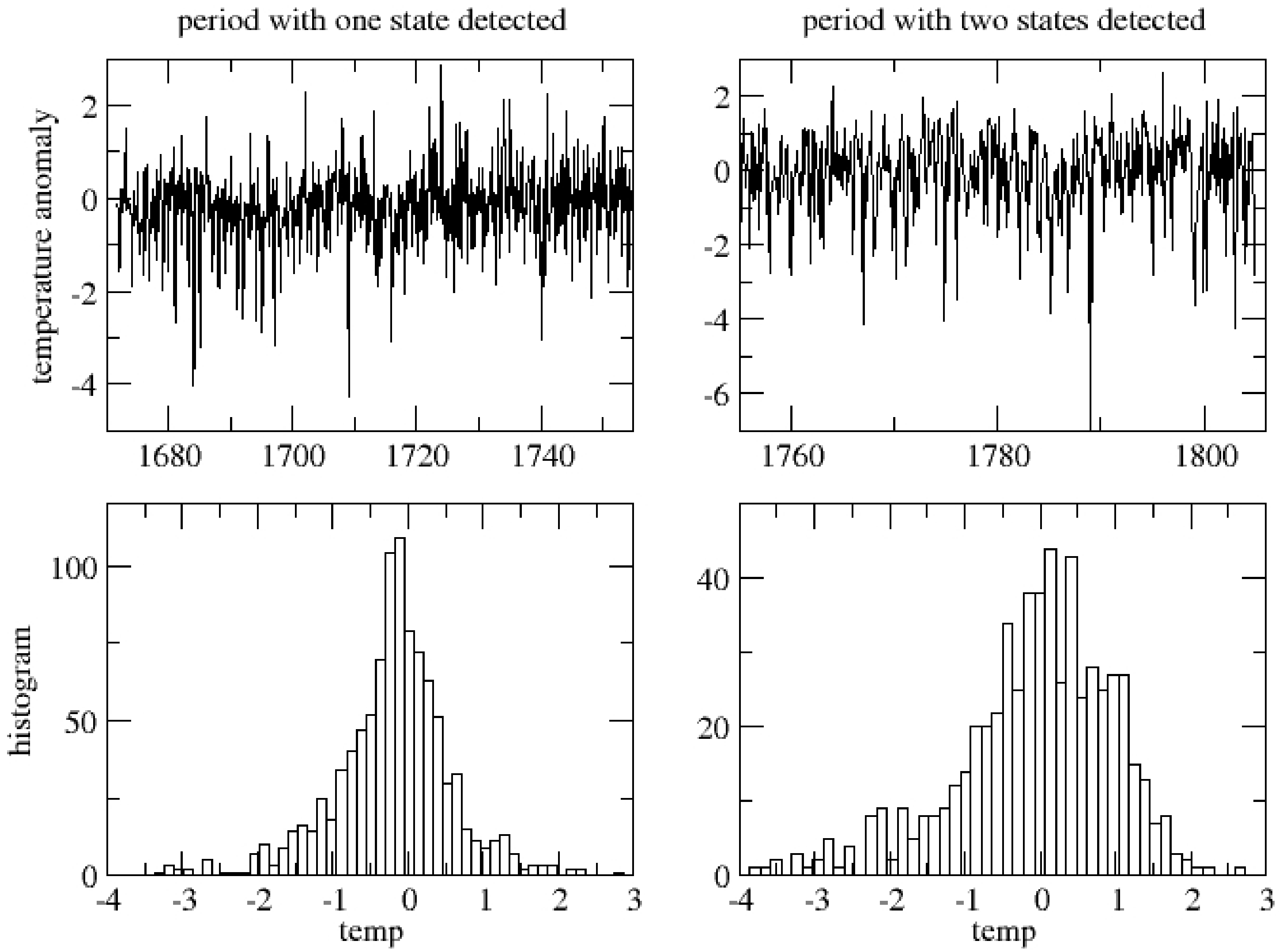



**A change of the number of states around year 1770. There were observed climatic anomalies in the second half of the 18th century named after Baron de Malda, who recorded observational notes, "Malda anomaly``, when *the Mediterranean climate was "strange", with thunderstorms, floods, droughts, and severe winters.* The potential analysis shows appearing instability of the climate, when another, colder state was about to appear, but later that stabilised and formed 1-well potential again.**

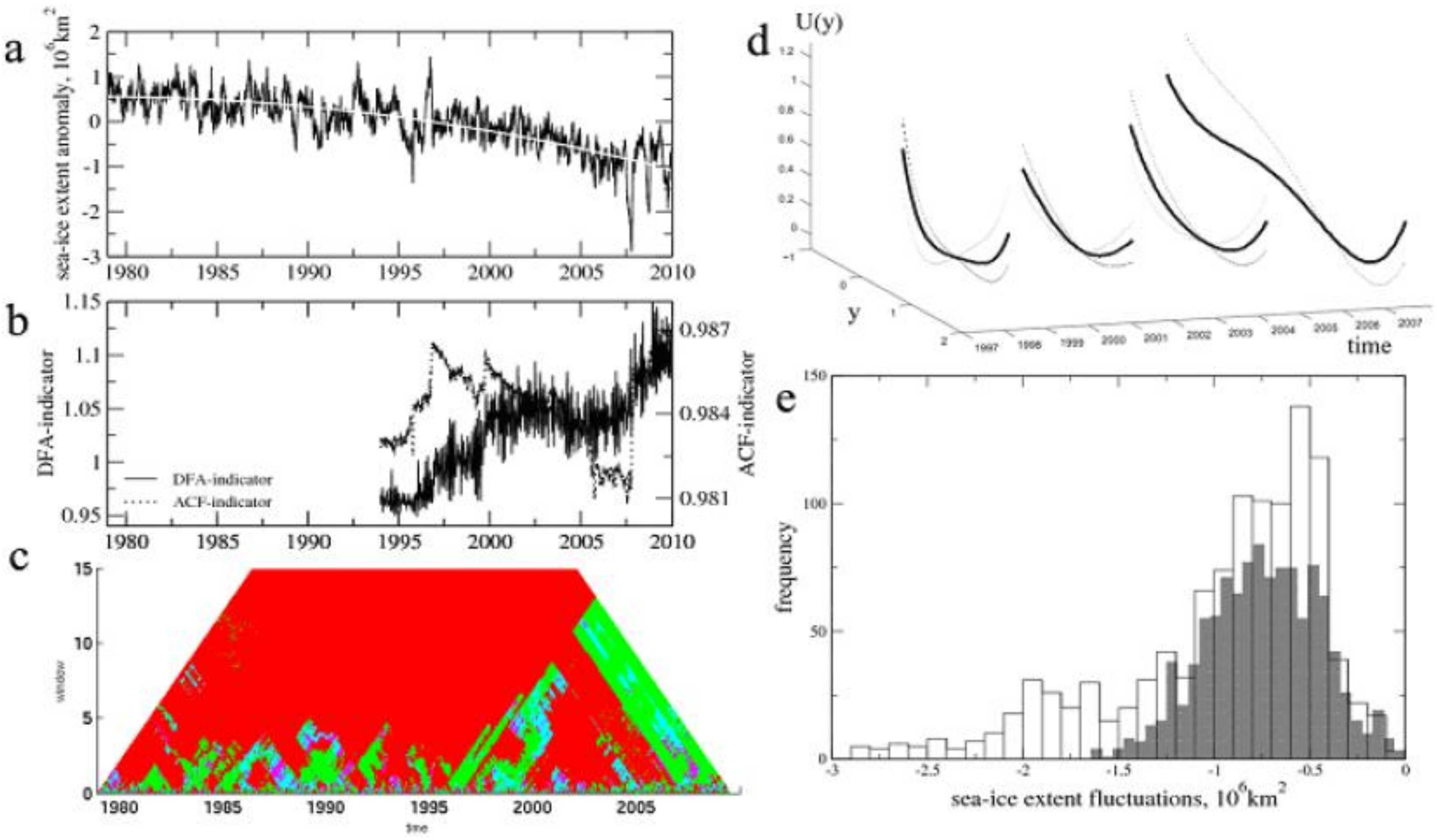
**Data:**
**Luterbacher et at, Science (2004)**
**Barriendos & Llasat, Climate change (2003)**

Livina et al, Climate Dynamics 2011

# European temperature anomaly: histograms



# Arctic sea-ice anomaly 1979-2011



# Preprocessing data for potential analysis

- When there are gaps or poor temporal resolution, we can interpolate data, because we deal with probability distribution. [To some extend: too high resolution together with small sliding window may give meaningless results].

- If there are nonstationarities in the data, it is helpful to use wavelet denoising for the estimation of the noise level and also detrending/filtering if there is an obvious trend

- Still, many datasets are studied in “raw” format

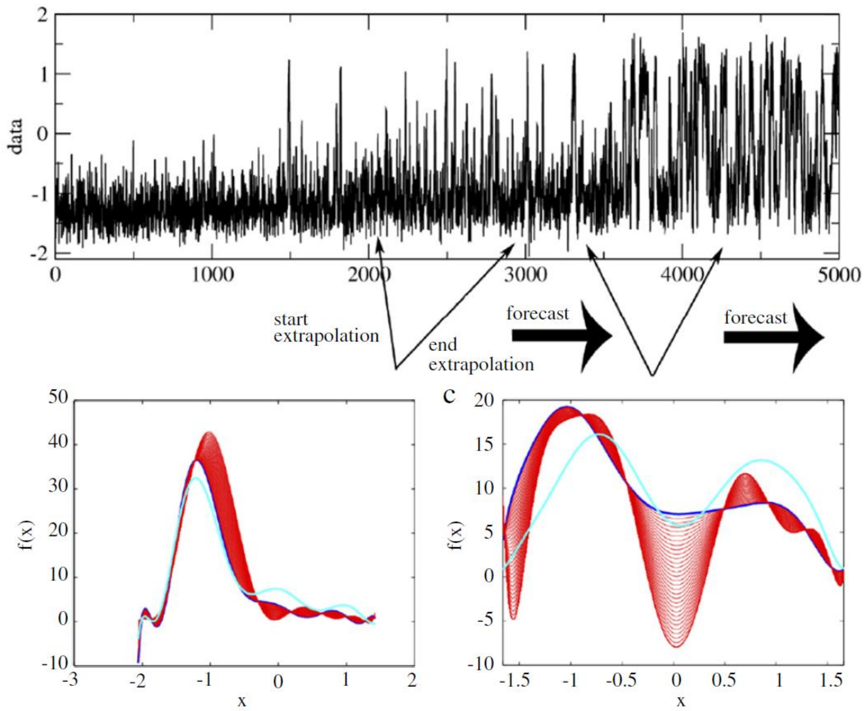
# Forecasting tipping points

# Potential forecasting algorithm

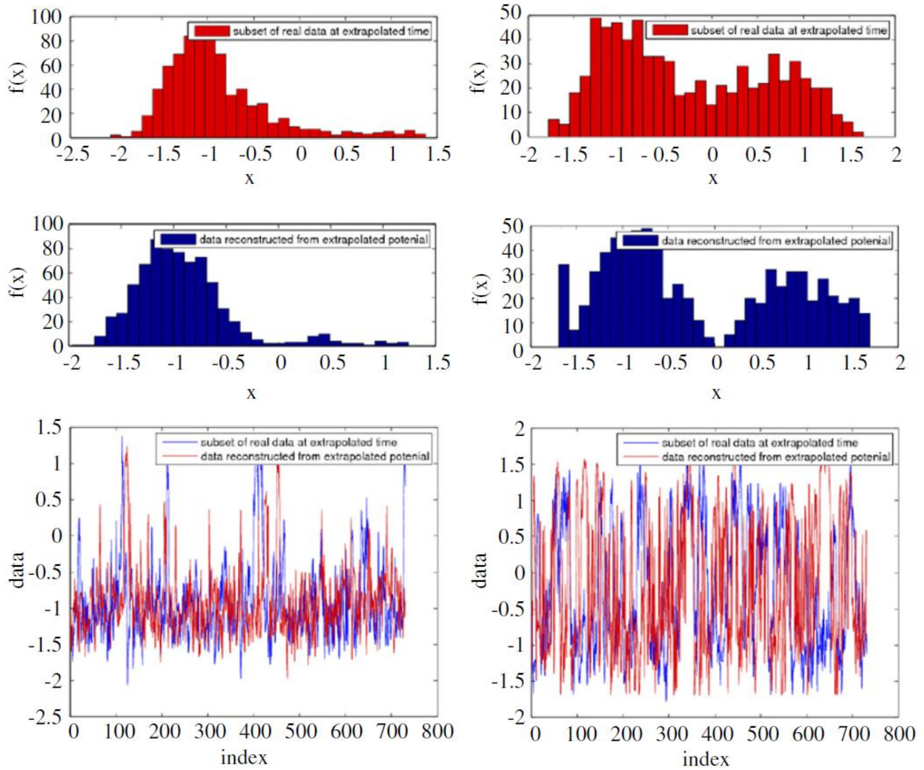
- Collect coefficients of Chebyshev approximation of PDF in sliding windows
- Extrapolate series of the coefficients
- Reconstruct forecast PDF
- Simulate time series from the obtained PDF (rejection sampling)
- Sort the series according to historic data (taking into account seasonality)

Livina et al, Phys A 2013

# Potential forecast of bifurcating artificial data



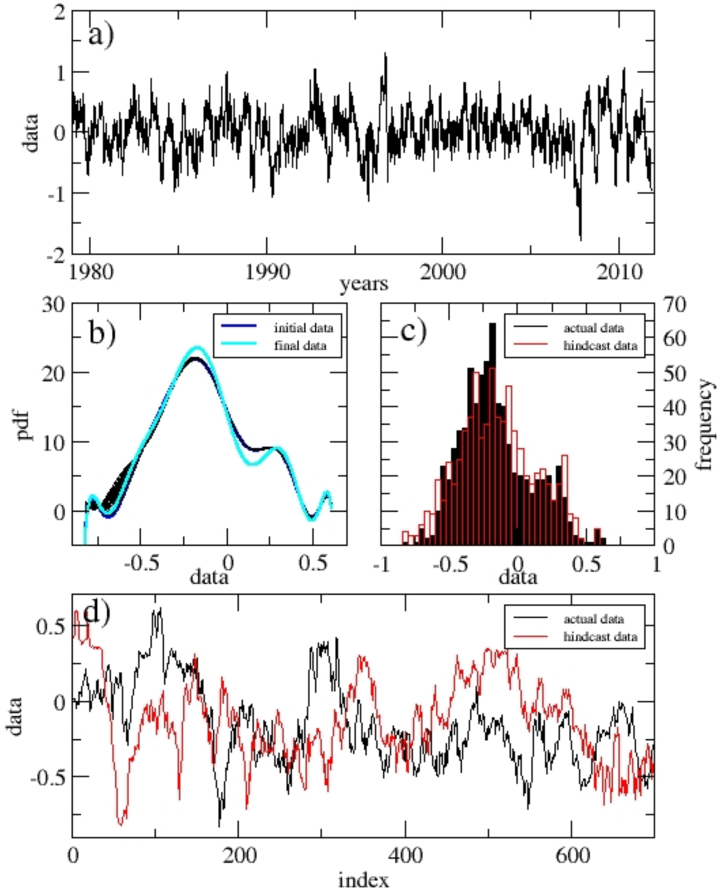

Two hindcasts shown: first collecting coefficients, then extrapolating and generating forecast time series



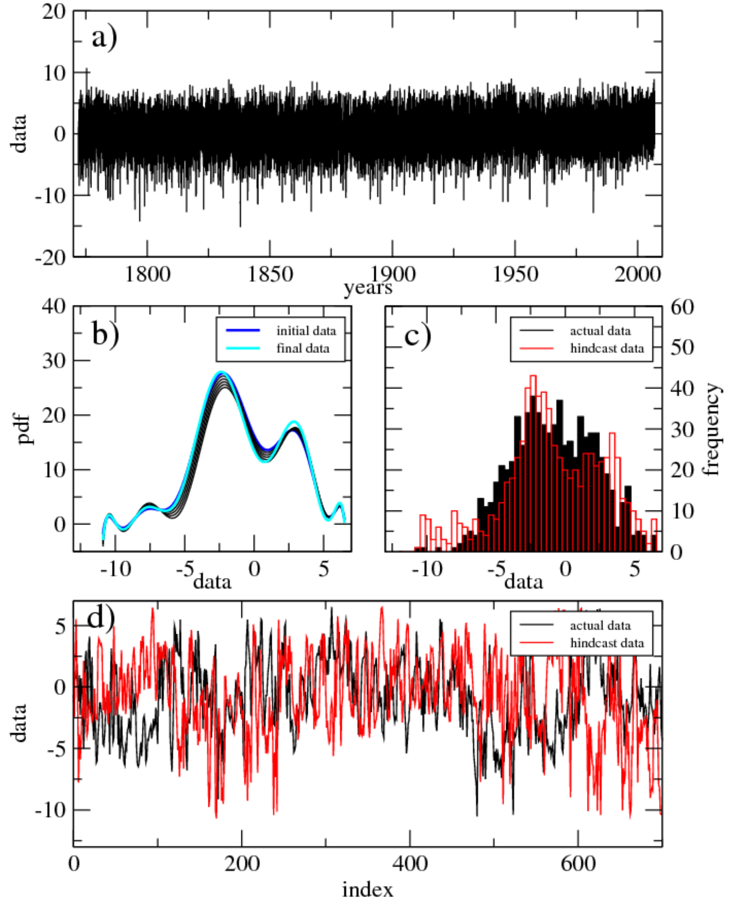

Extrapolated probability densities, histograms after extrapolation (actual and forecast) and time series for comparison (actual and forecast)

# Potential forecasting: geophysical data

Livina et al, Physica A, 2013



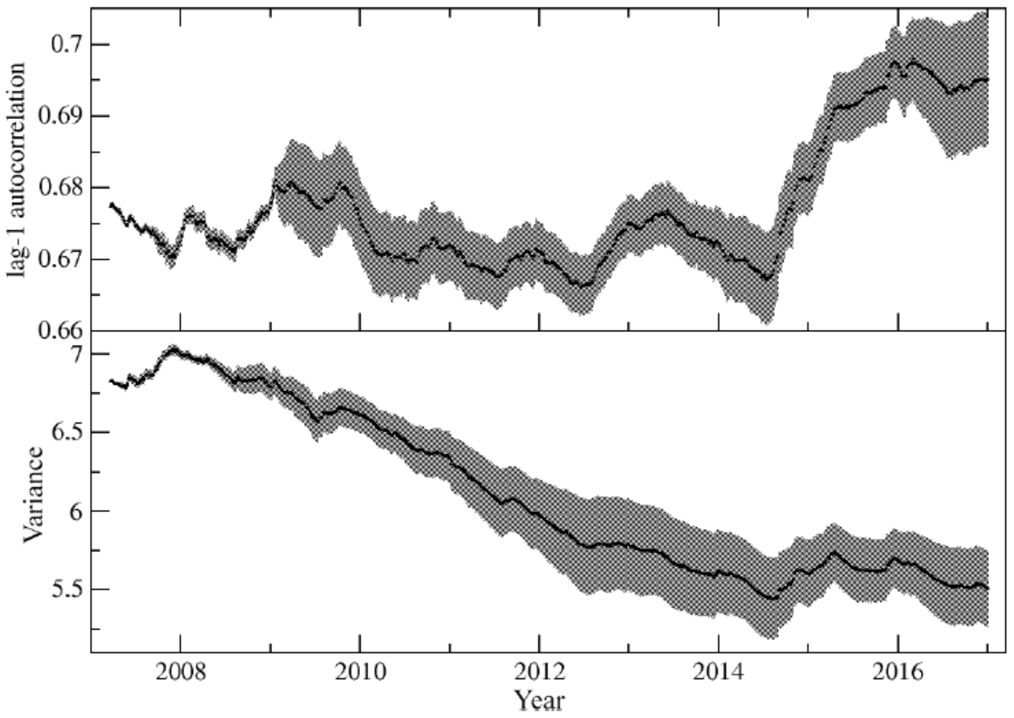

Arctic sea-ice extent




Central England temperature

# Various applications

# Deep ocean acoustic noise data



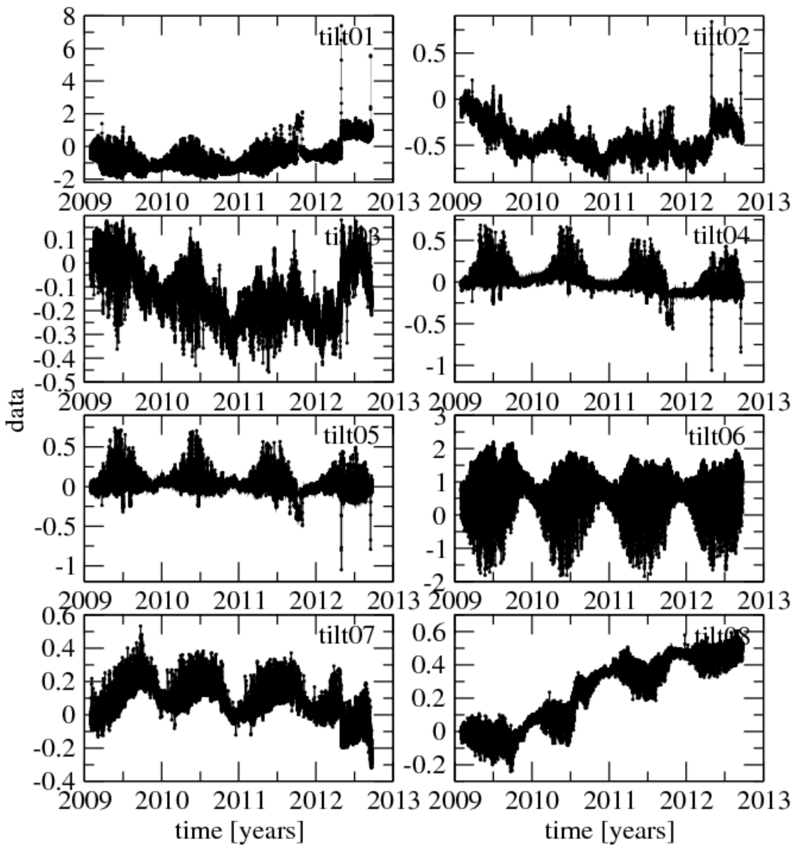

EWS of the anomaly 2015-2016: acoustic data at 1km depth near Australia has a temperature signature of El Nino! SOFAR acoustic channel allows sound propagate for many kilometres

- Database of the Preparatory Commission for the Comprehensive Nuclear-Test-Ban Treaty Organisation (CTBTO)
- Cape Leeuwin hydrophone, series H01W1 (Australia)
- Long record (since 2003), 250Hz sampling of sound pressure
- 3Tb of binary waveforms, 35Tb of extracted signal, 96G points in time series
- 10-minute averages of sound pressure level (SPL) in five frequency bands: 10-30, 40-60, 85-105, 5-115 and 56-70 Hz; giving 630K points per time series

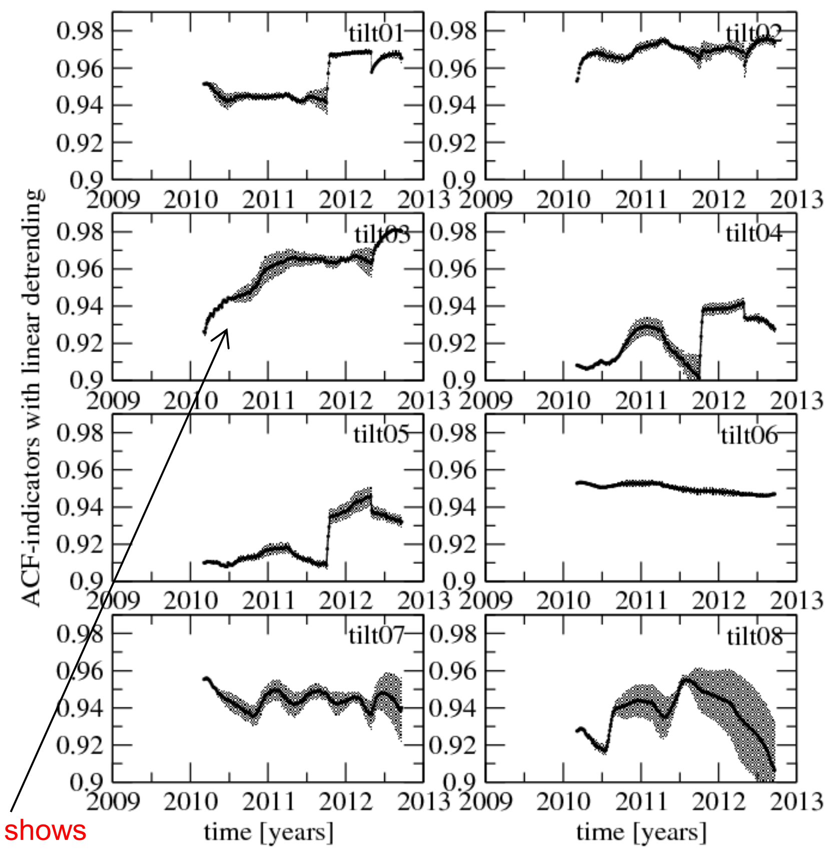
# Structure health monitoring:

## NPL footbridge data

(Livina et al, JCSHM 2014)



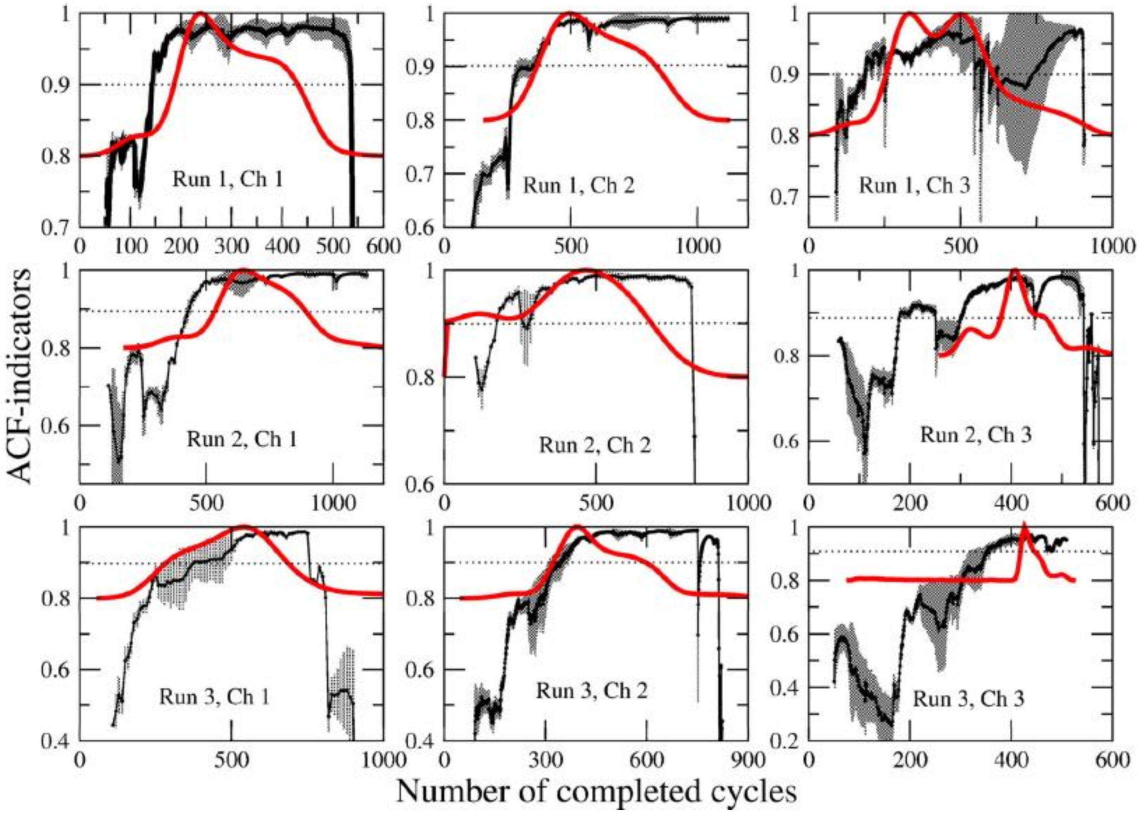

data

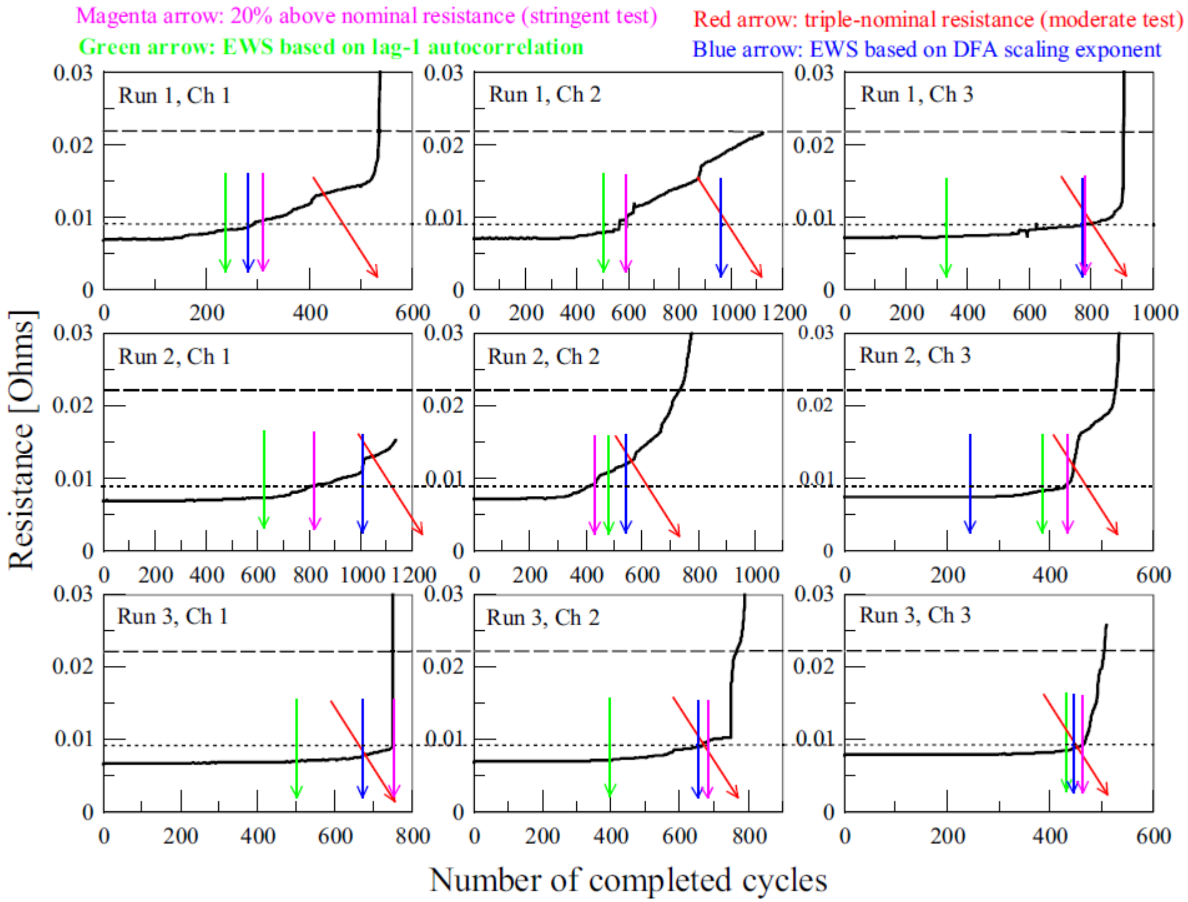



Sensor 3 shows most vulnerable bridge part

indicators

# Tipping in electromagnetic data (aging components)

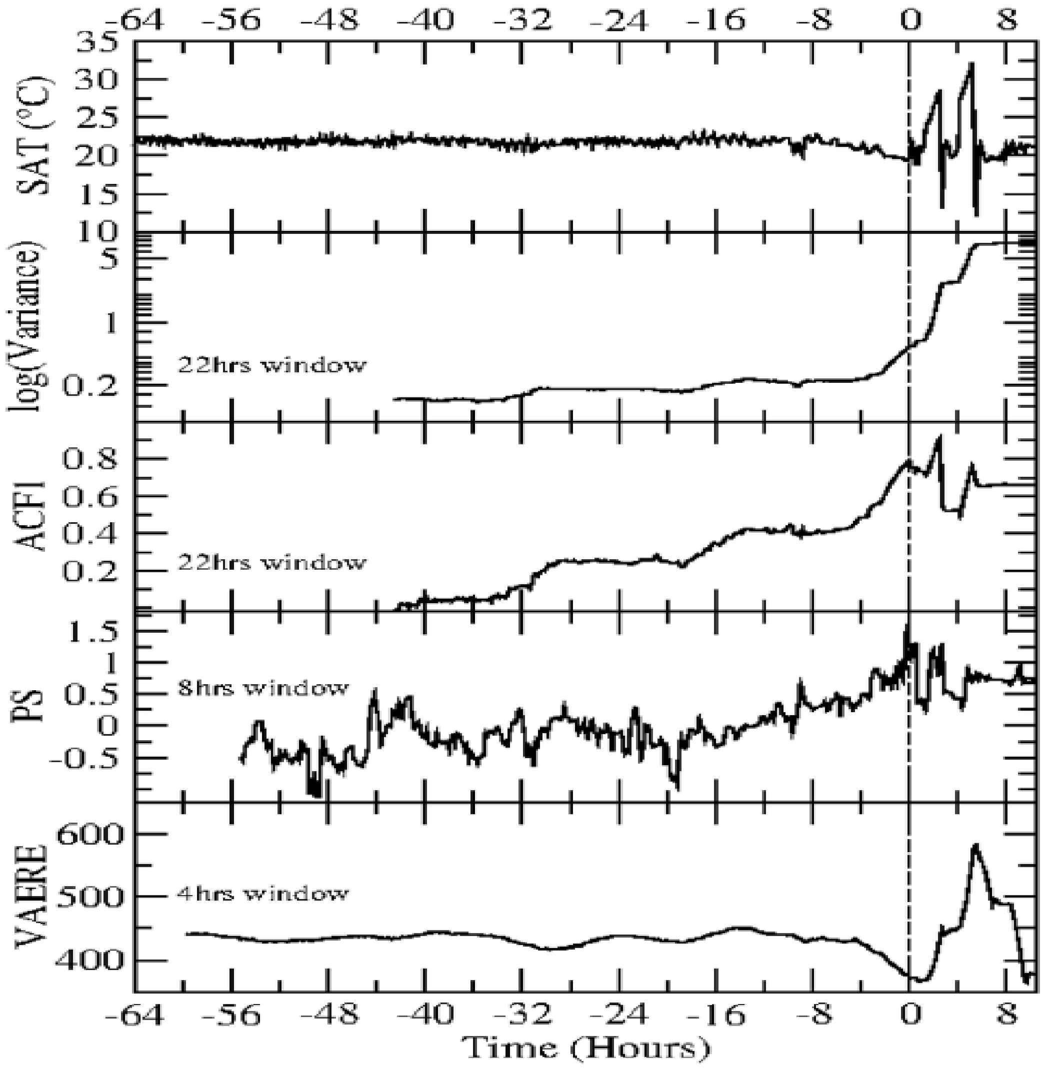



EWS indicators are extrapolated to estimate when in future they would reach critical value 1 - this provides a set of possible times when the failure would happen, which forms a histogram that is used to generate the kernel density of the future failure times. The peak of such a kernel density is the most likely time of failure.

Livina et al, JET 2020

# Sensor data in built environment




Building Management System (BMS): sensor data gathered in January 2019, with a failure of a ventilation valve that led to abnormal thermal oscillations in the building.

Using this data, we detected early warning signal of valve failures; this algorithm was implemented for real-time data monitoring in the control room of Mitie company.

Mesa et al, IJMQE 2021

# WISE:
# Water Information System for Europe

Water Framework Directive 2000/60/EC committed the EU member states to achieve good qualitative and quantitative status of all water bodies by 2015.

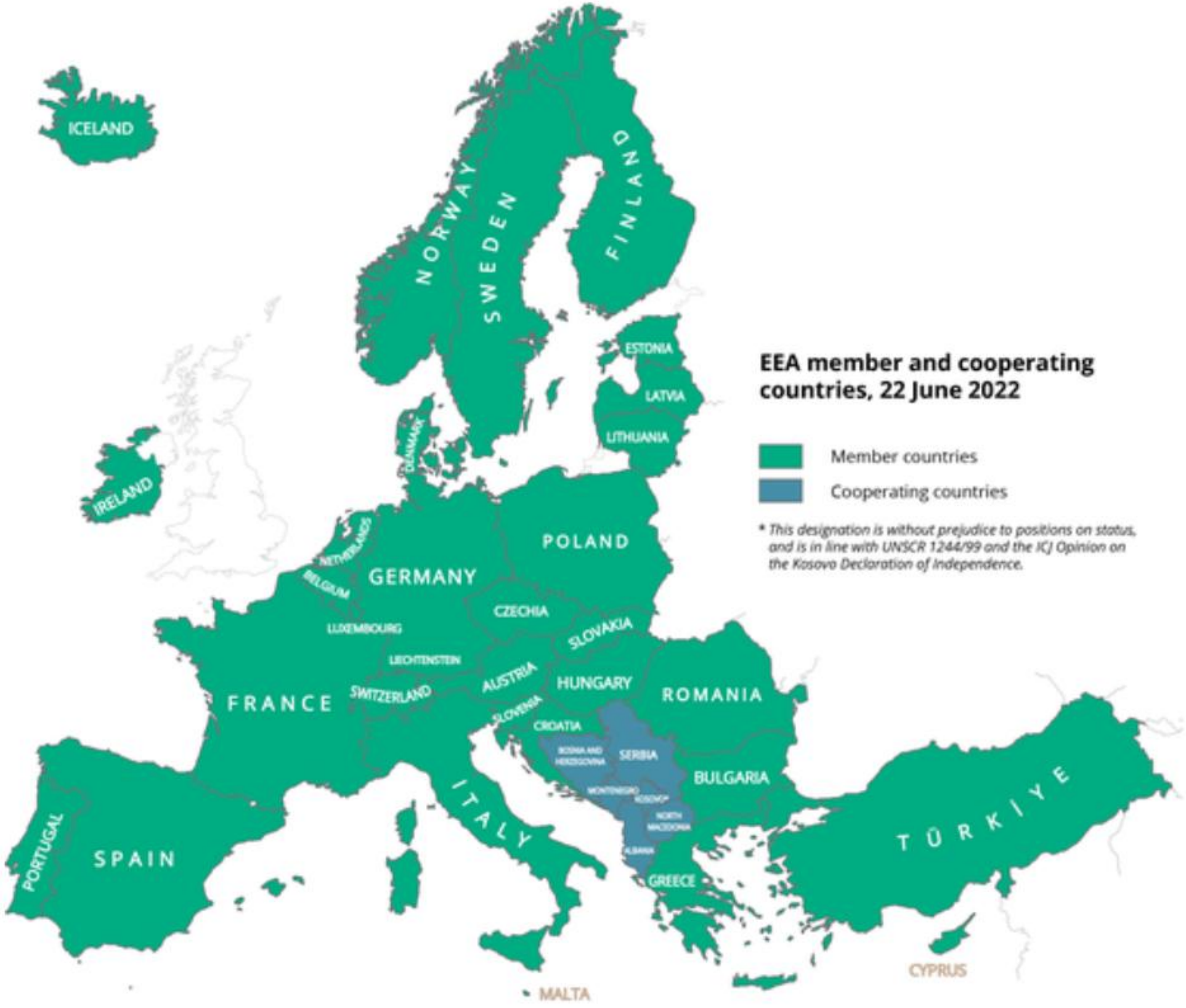
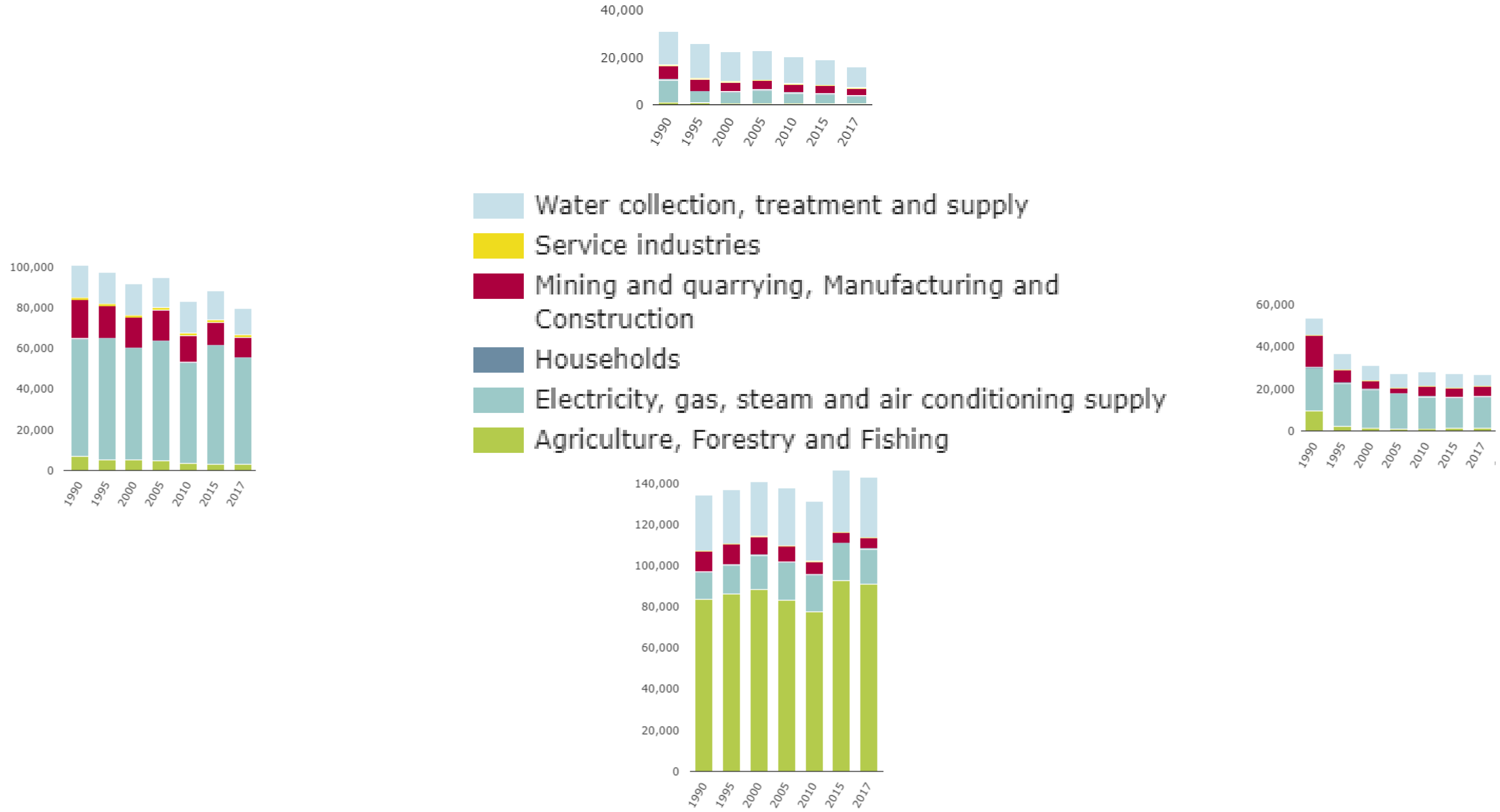



Figure: courtesy of WISE (https://water.europa.eu/)

# Regional freshwater use (abstraction)

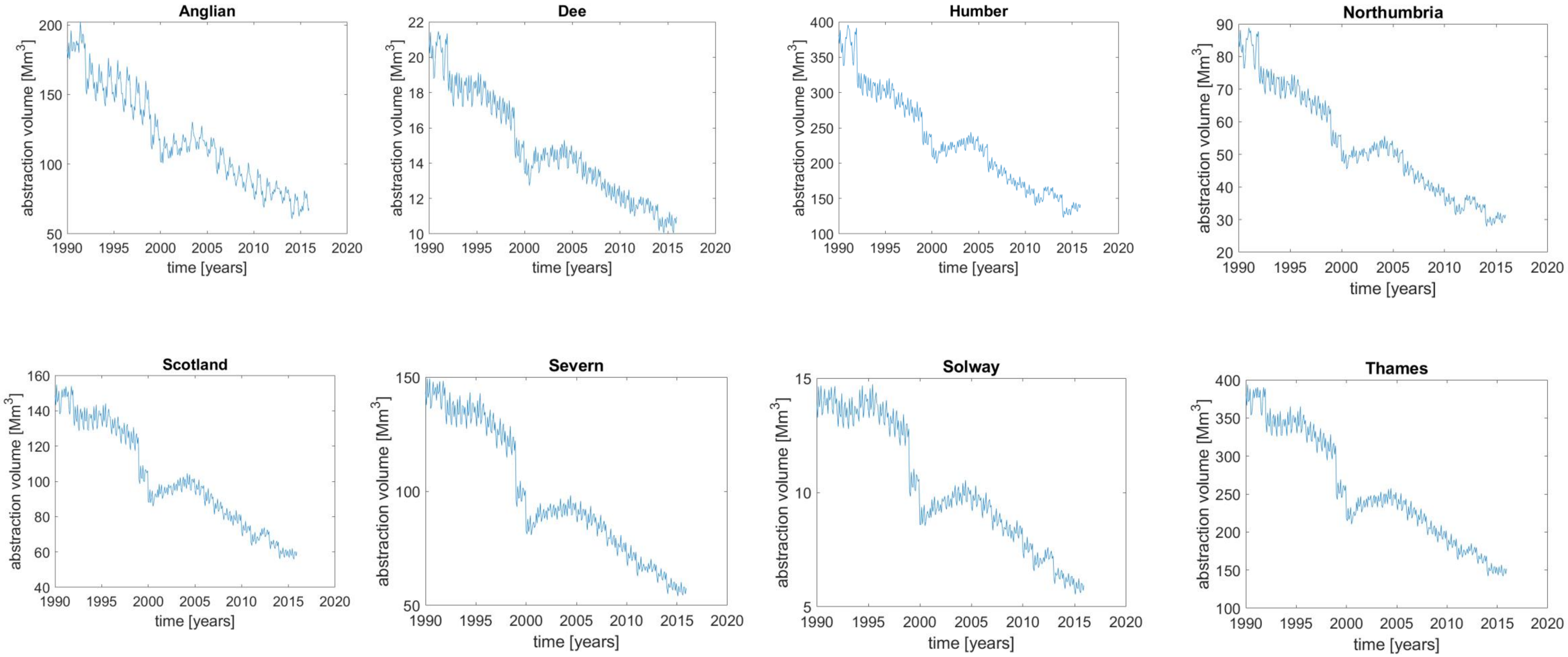


# European basins: improving or worsening?


## Regions with increasing water abstraction

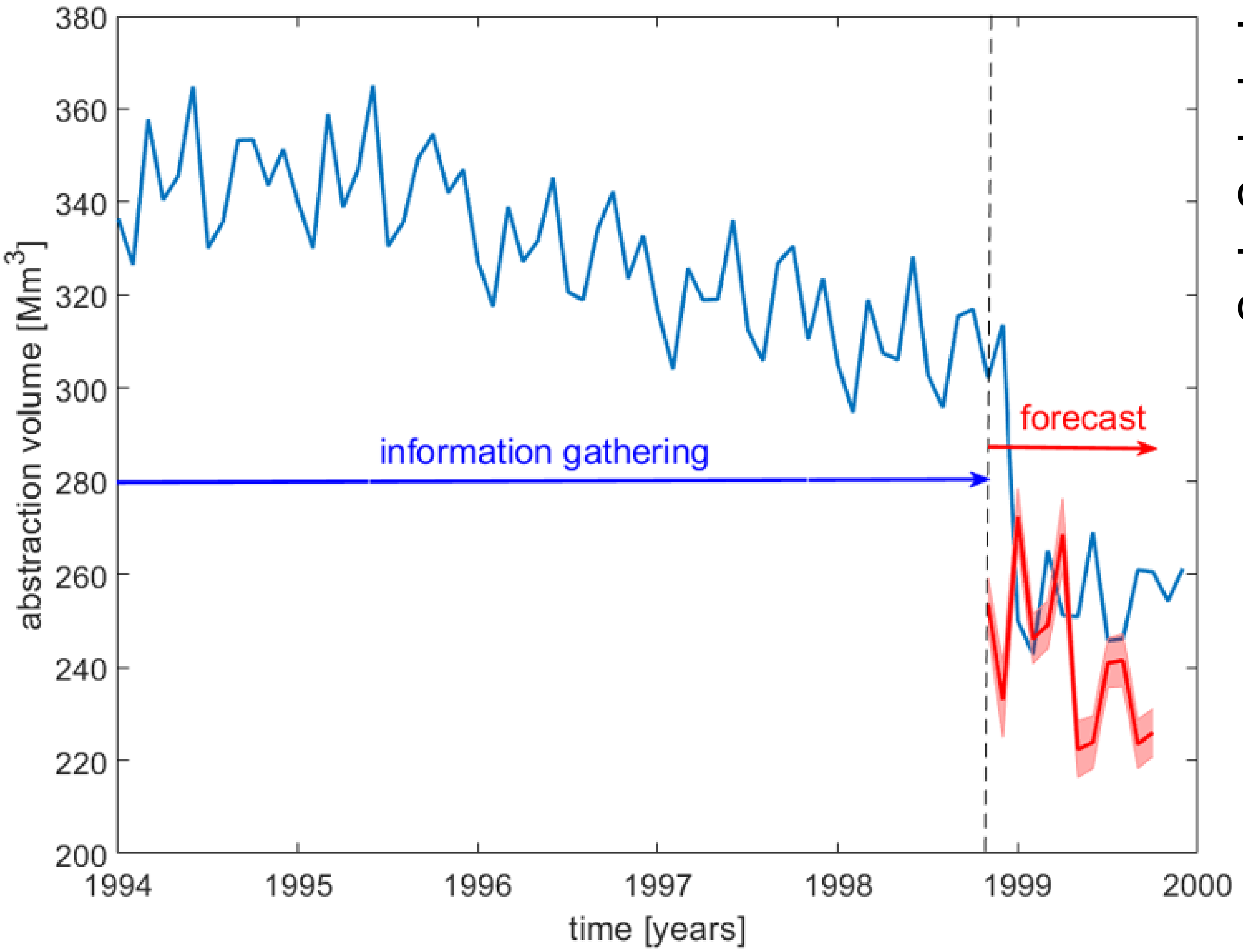
- Peloponnese (dry region + agri)
- Iberian (dry region + agri)
- Finland (abundant water supply yet increasing water use)
- Ireland (sufficiently supplied, increasing use)

# UK

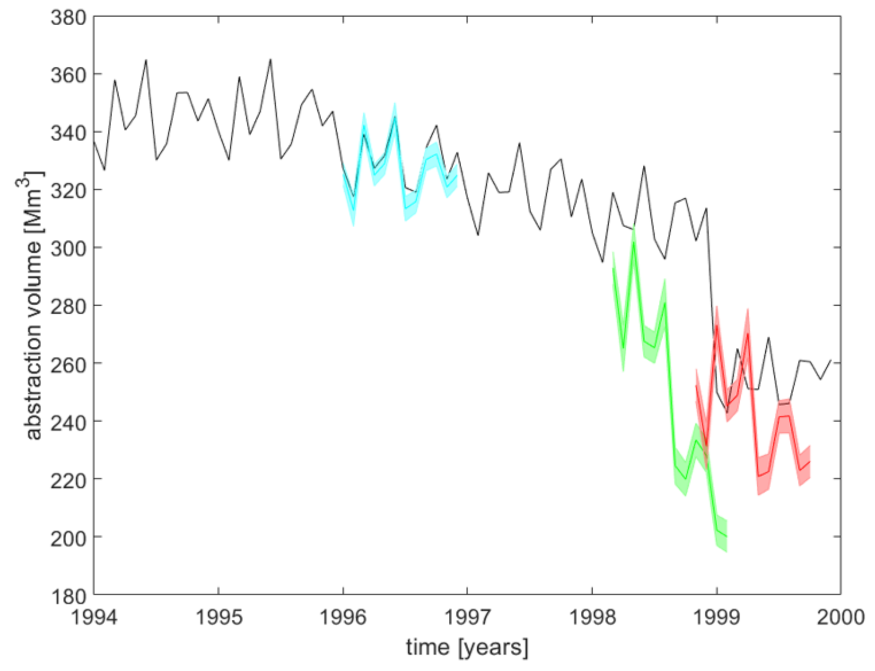



The groundwater drought of 1995-1998 - further reform of the regulatory framework: parliamentary debate highlights the environmental impacts of drought and the Parliament grants new powers to the Environment Agency to issue drought permits.

# The Thames potential hindcast of drought




Gather information in subsets:
- Linear trend
- Seasonal sine-like trend
- Fluctuations, with uncertainty quantified over 20 samples
- The result is sensitive to the cut-off point of data gathering



# ML/AI developments for tipping points

- A new field with many publications already – use techniques for anomaly detection and forecast
- Often require tuning of multiple parameters
- Sometimes aggregate multiple EWS indicators
- Risk of overfitting and performing well only on specific data
- Uncertainty quantification may be affected by NN architecture
- Heavier computationally than the original EWS techniques

# Summary

NPL

- Three stages of the tipping point analysis: anticipating of tipping points using early warning signal indicators; detecting tipping points using potential analysis; potential forecasting of tipping points
- Developed several novel techniques, tested on artificial data
- Applied to various datasets and sensor records of diverse dynamic systems
- Publications on tipping point analysis:


1) Livina & Lenton, GRL 2007
2) Lenton et al, PhilTrans RoyalSoc 2009
3) Livina et al, Climate of the Past 2010
4) Vaz Martins et al, Phys Rev E 2010
5) Livina et al, Climate Dynamics 2011
6) Lenton et al, PhilTrans RoyalSoc 2012
7) Lenton et al, CoP 2012
8) Livina et al, Physica A 2012
9) Dakos et al, PLOS ONE 2012
10) Livina and Lenton, Cryosphere 2013
11) Cimatoribus, CoP 2013
12) Drijfhout et al, PNAS 2013
13) Livina et al, Physica A, 2013
14) Livina et al, JCSHM 2014
15) Kefi et al, PLoS ONE 2014
16) Livina et al, Chaos 2015
17) Perry et al, SMS 2016
18) Lenton & Livina, AGU 2016
19) Perry et al, SMS 2016
20) Livina et al, NPG 2018
21) Prettyman et al, EPL 2018
22) Prettyman et al, Chaos 2019
23) Livina et al, JET 2020
24) Mesa et al, IJMQE 2021
25) Prettyman et al, ERL 2022
26) Billuroglu & Livina, JFAP 2022
27) Livina, Nature Climate Change 2023
28) Radhakrishnan et al, Nature Scientific Reports 2025
29) Livina, Chaos 2026
30) Livina et al, in preparation

# The Fourth Paradigm:
## data-driven discovery

- ➢Empirical science (1000s years)
- ➢Theoretical science (100s years)
- ➢Computational science (10s years)
- ➢Data exploration science (today)

http://research.microsoft.com/en-us/collaboration/fourthparadigm/

**Stochastic modelling of complex systems is becoming more important: generic & lightweight**

# Discussion

- Analytical solutions for time-varying potentials in complex systems? Or numeric solutions only?
- Data-driven separation of deterministic and stochastic components?

# Acknowledgements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# Thank you