STOCHASTICA

Stochastic fluid dynamics & skew-symmetric multiplicative noise

Valentin Resseguier & many others

INRAE
FREIE UNIVERSITÄT BERLIN
IMT Atlantique Bretagne-Pays de la Loire École Mines-Télécom
Ifremer
Inria
Imperial College London
BROWN UNIVERSITY
河海大學 HOHAI UNIVERSITY

Content

I. Some issues in fluid dynamics
- Learning turbulent systems dynamics
- Bayesian inverse problems

II. Transport noise
- Stochastic Lagrangian models
- Stochastic Navier-Stokes equations under location uncertainty (LU)
- Broader scope on transport noise (SALT, LU, …)

III. Multiplicative score-based generative model
- Big picture
- Dynamics on d-spheres & Fokker-Planck equations
- Latent space
- Backward diffusion with a neural network & ELBO
- Numerical results
- Open questions

IV. Beyond classical transport noises
- Scale symmetry
- Fractals in space and time
- Gaussian Multiplicative Chaos
- Open questions

I. Some issues in fluid dynamics

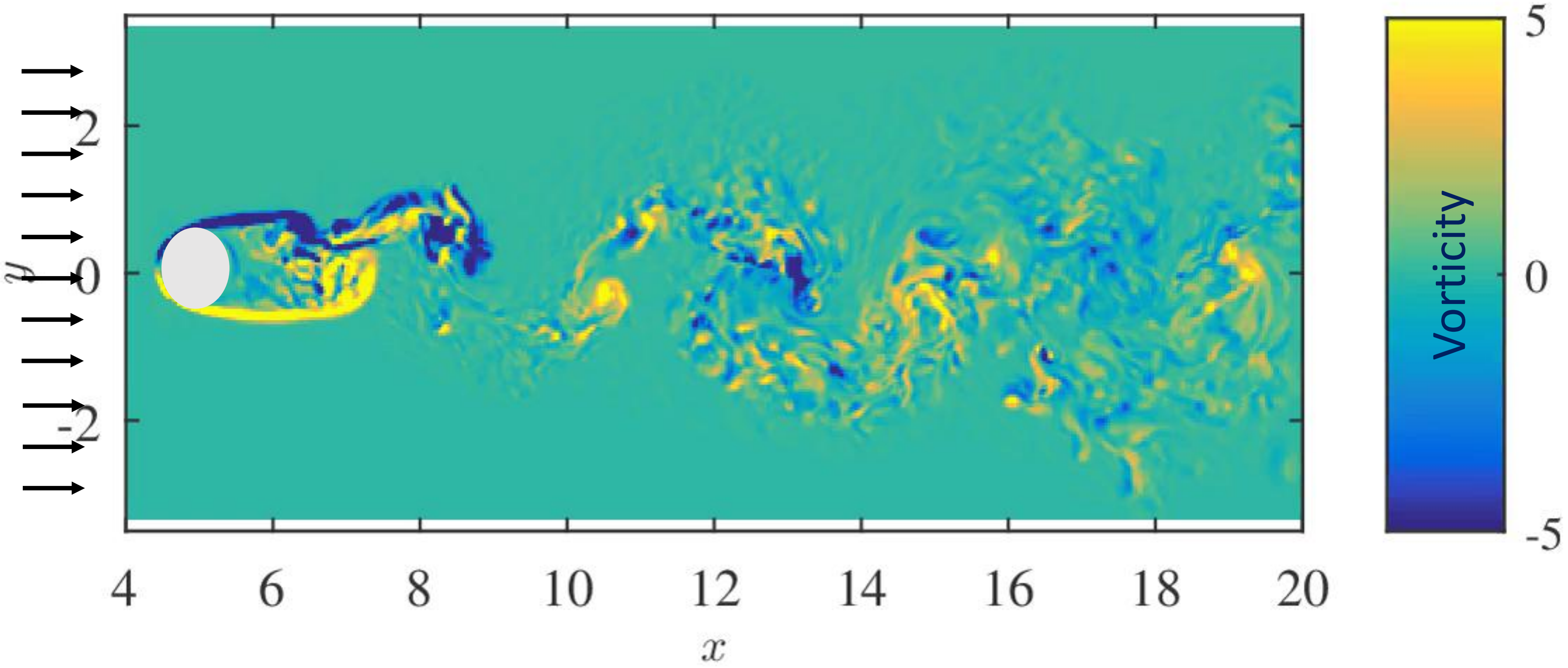
- Learning turbulent systems dynamics
- Bayesian inverse problems

RÉPUBLIQUE FRANÇAISE
Liberté
Égalité
Fraternité

INRAE

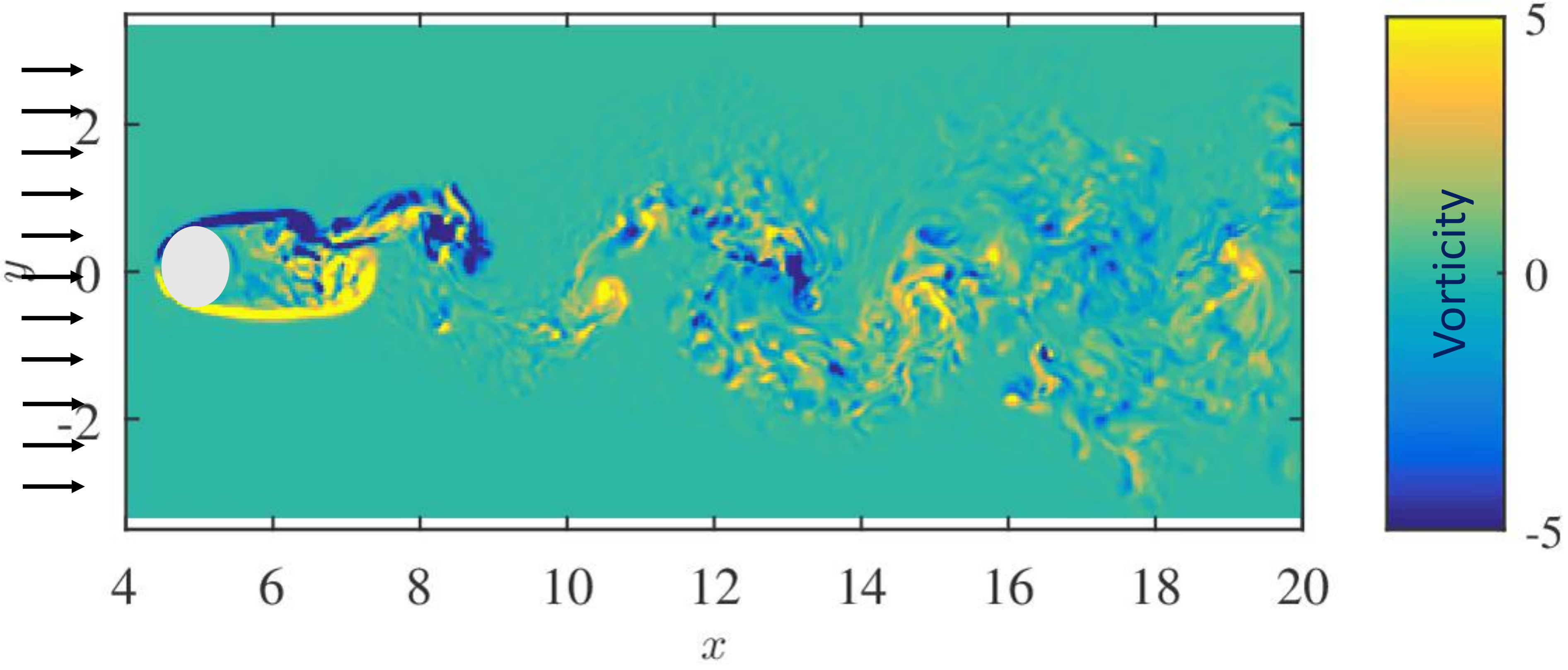
Issues when learning turbulent systems dynamics

1) Fluids are multiscale → Many coupled degrees of freedom → **Curse of dimensionality** / Slow Kolmogorov n-width decay



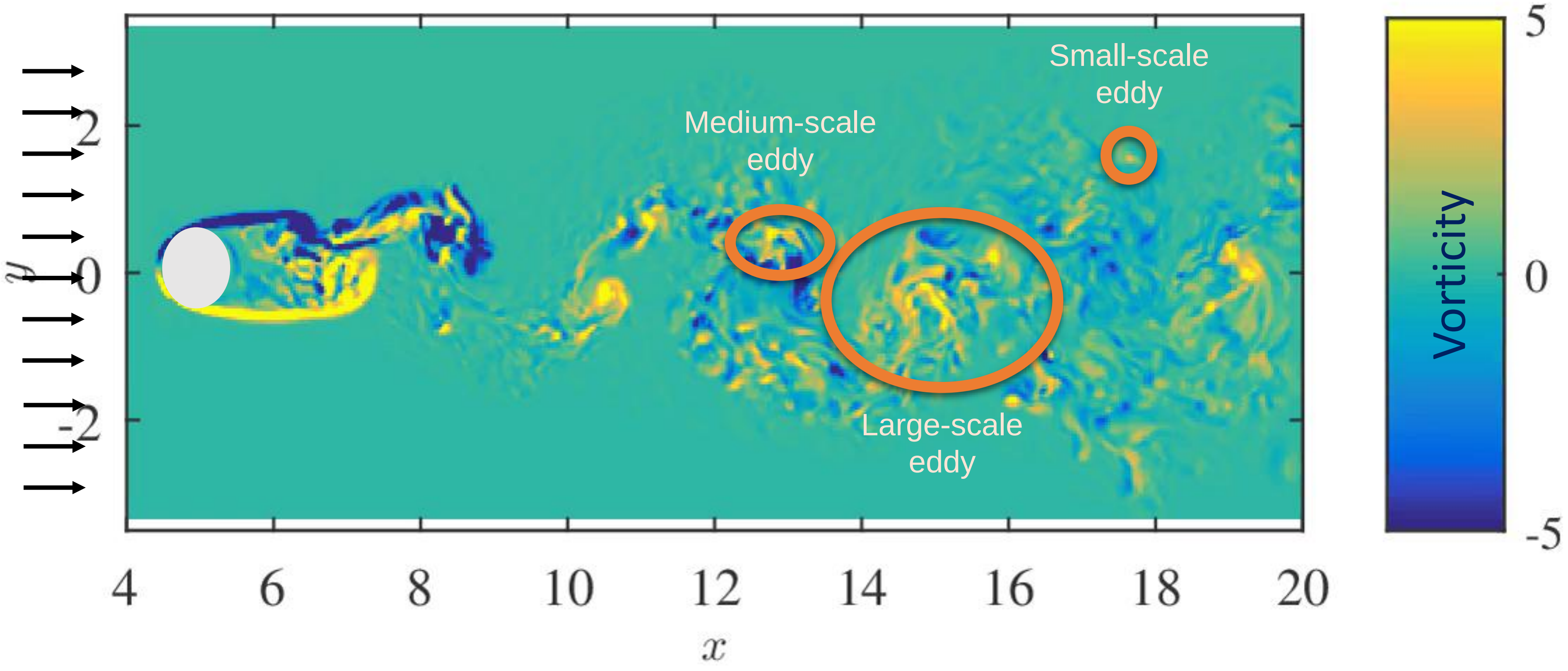
Issues when learning turbulent systems dynamics

1) Fluids are multiscale → Many coupled degrees of freedom → **Curse of dimensionality** / Slow Kolmogorov n-width decay



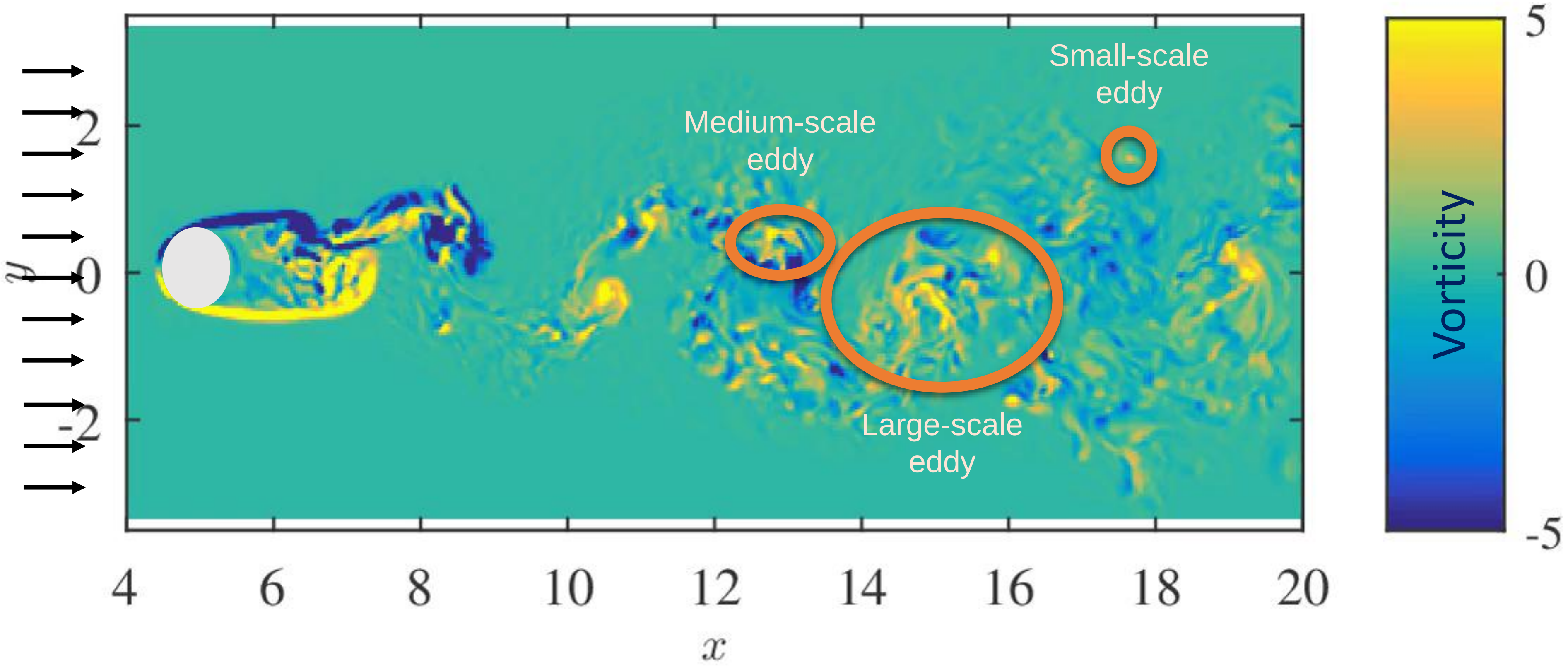
Issues when learning turbulent systems dynamics

1) Fluids are multiscale → Many coupled degrees of freedom → **Curse of dimensionality** / Slow Kolmogorov n-width decay



Issues when learning turbulent systems dynamics

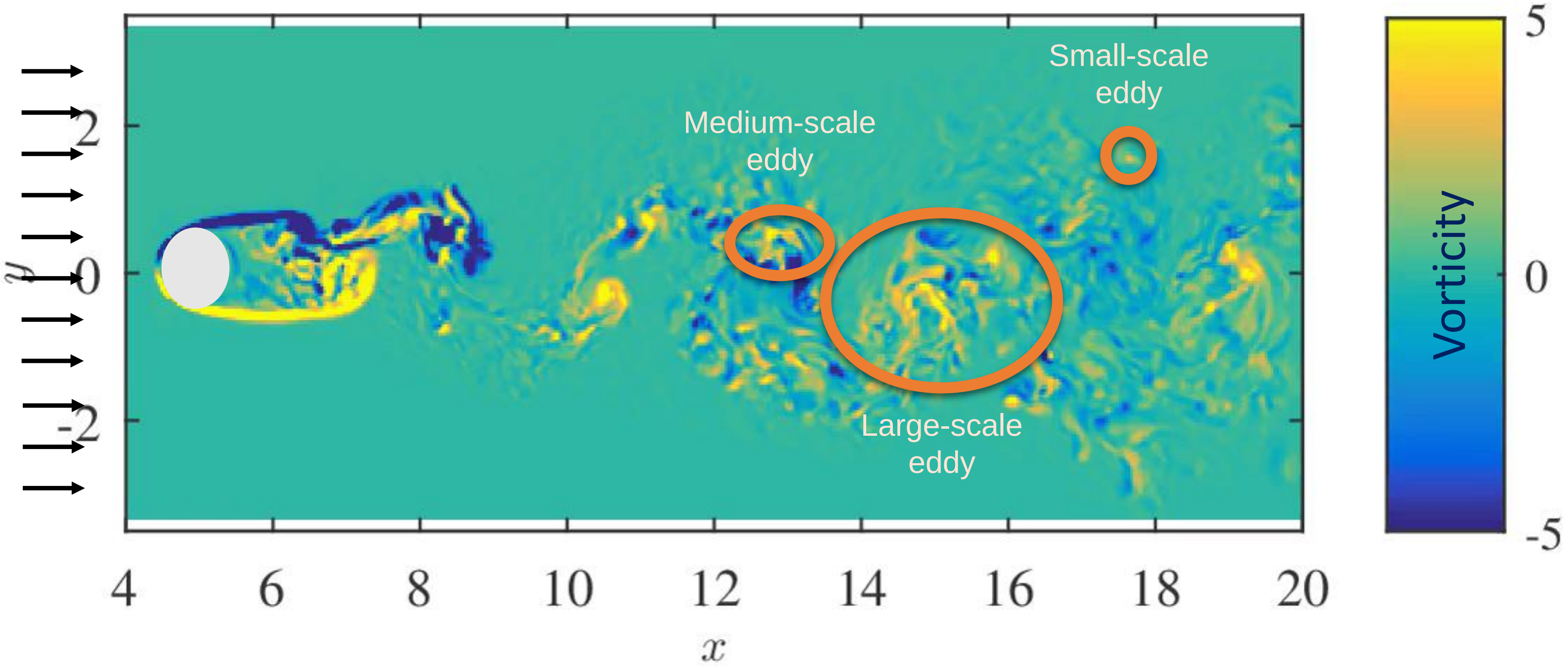
1) Fluids are multiscale → Many coupled degrees of freedom → **Curse of dimensionality** / Slow Kolmogorov n-width decay



2) Fluids are intermittent → **Varying 2nd order statistics + Extremes** → Huge data set needed

Issues when learning turbulent systems dynamics

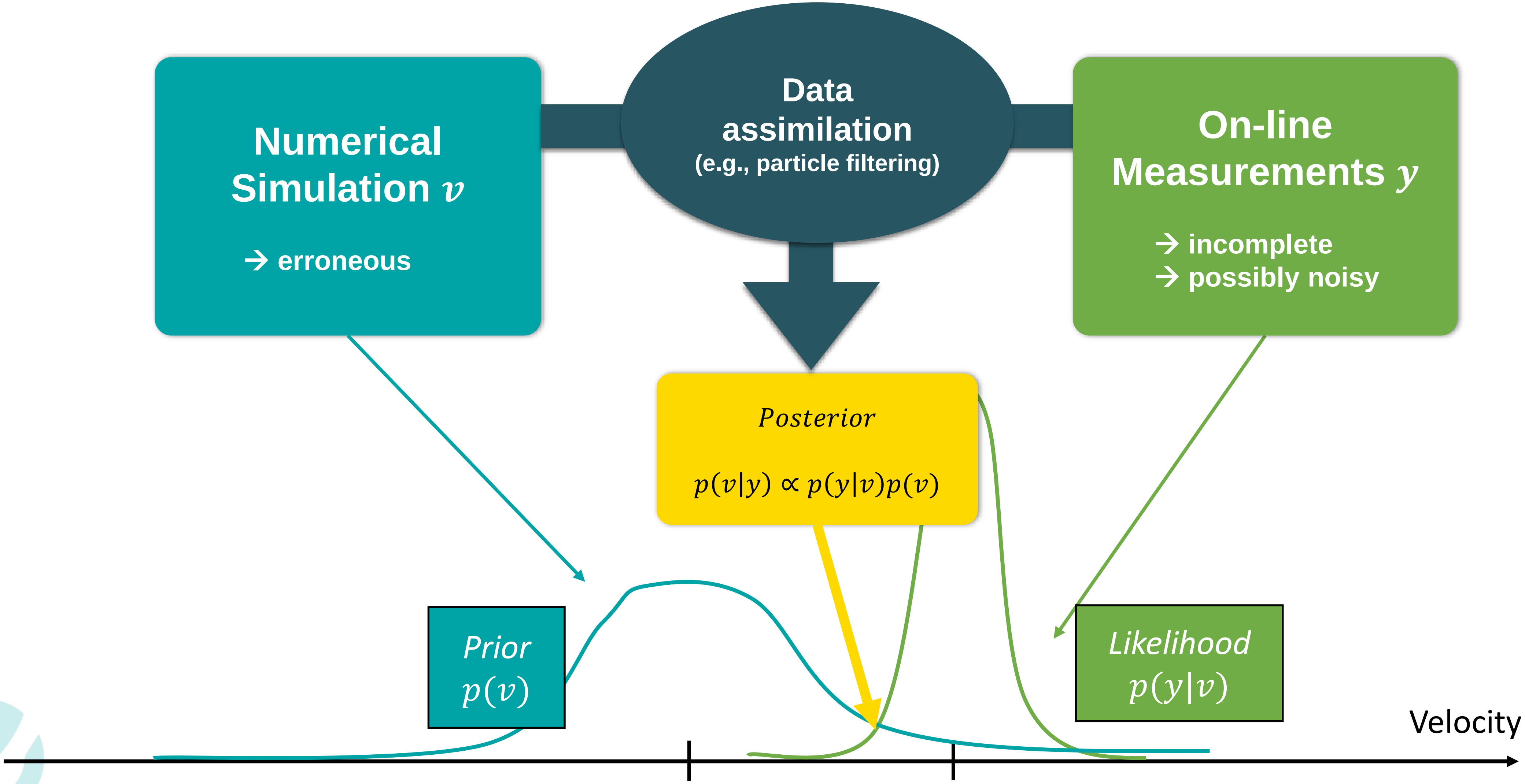
1) Fluids are multiscale → Many coupled degrees of freedom → **Curse of dimensionality** / Slow Kolmogorov n-width decay



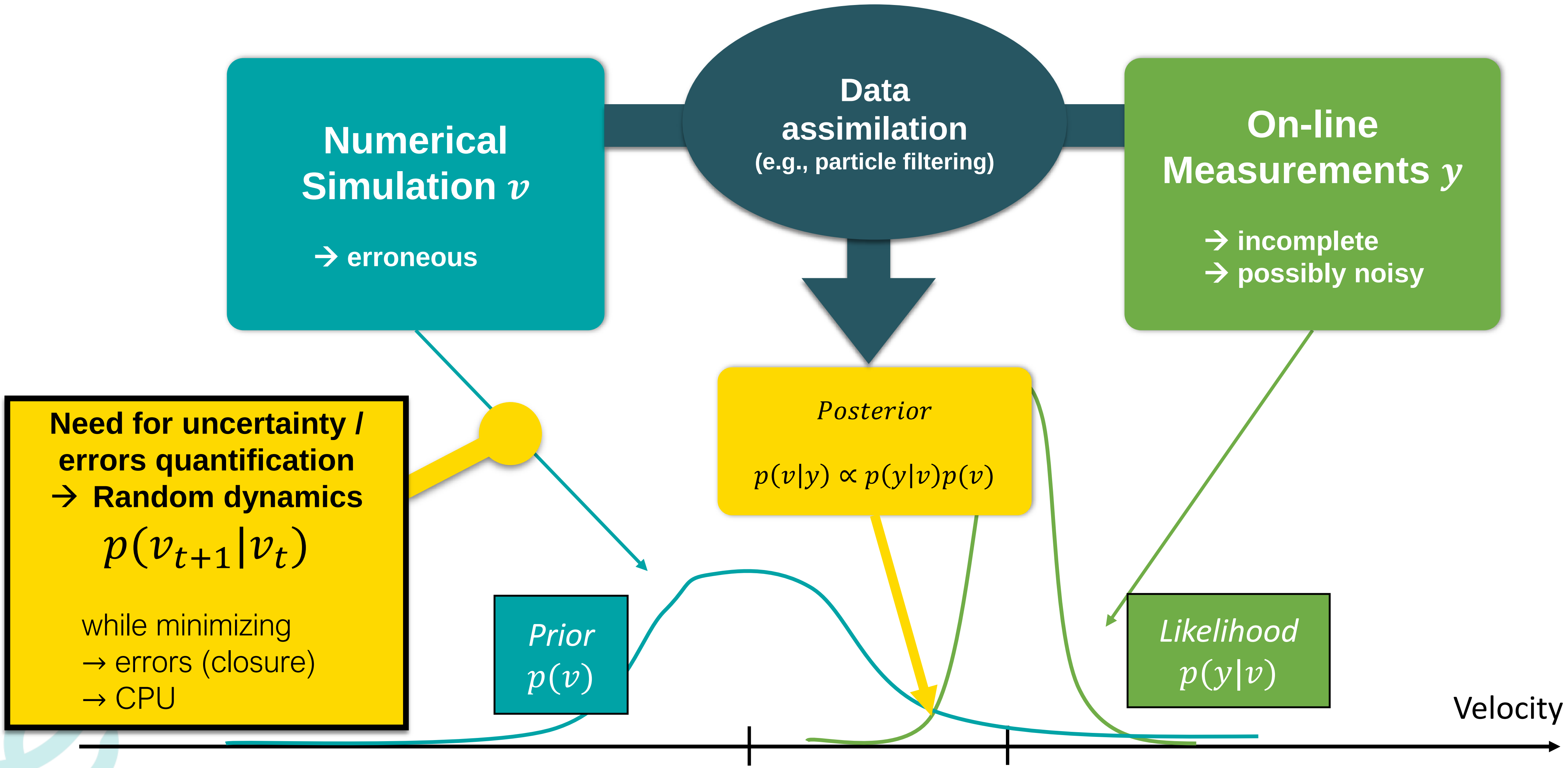
2) Fluids are intermittent → **Varying 2nd order statistics + Extremes** → Huge data set needed

3) Fluids are chaotic → Small errors quickly increases → **Robustness issues** for unsteady problems

Issues in Bayesian inverse problems / filtering / data assimilation



Issues in Bayesian inverse problems / filtering / data assimilation



II. Transport noise

- Stochastic Lagrangian models
- Stochastic Navier-Stokes equations under location uncertainty (LU)
- Broader scope on transport noise (SALT, LU, …)

In collaboration with : Wei Pan, Yicun Zhen, Bertrand Chapron, Baylor Fox-Kemper, Etienne Mémin





A famous stochastic fluid dynamic
Stochastic Lagrangian models (SLM) (≠ transport noise)

$v = u + v'$

Resolved fluid velocity:
u

Unresolved Forces:
$F_\sigma(t, X_t) \circ \frac{dW_t}{dt}$
increments of **Wiener** process
(Gaussian, white wrt t)

Momentum conservation

$$d\big(u_t(X_t)\big) = F(t, X_t, u_t)dt + \underbrace{F_\sigma(t, X_t, u_t) \circ dW_t}_{} \quad \text{(Forces)}$$

conditionnally
Gaussian
process
white in time
and in space

Positions of fluid parcels X_t :

$$dX_t = u_t(X_t)dt + 0$$

Pope (1983, 1994), Pope & Chen (1990)
Bossy & al. (2015,2022)

Another approach : Transport noise
and 1st open and question : *"The £10 problem"*

$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:
$$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt}$$
$$:= \sigma(x) \circ \frac{dB_t}{dt}$$
increments of **Q-Wiener** process
(Gaussian, white wrt t)

$$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt} \right) = \nabla \cdot u^S = 0$$

Another approach : Transport noise
and 1st open and question : *"The £10 problem"*

$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:
$$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt}$$
$$:= \sigma(x) \circ \frac{dB_t}{dt}$$
increments of **Q-Wiener** process
(Gaussian, white wrt t)

$$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt}\right) = \nabla \cdot u^S = 0$$

Newton 2nd law

Momentum conservation
$$\text{“}d(dX_t) = \text{Forces}(t, X_t, u_t)\text{”}$$

Positions of fluid parcels X_t :
$$dX_t = u_t^S(X_t)dt + \sigma_t(X_t) \circ dB_t$$

Another approach : Transport noise
and 1st open and question : *"The £10 problem"*

$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt}$$
$$:= \sigma(x) \circ \frac{dB_t}{dt}$$

increments of **Q-Wiener** process
(Gaussian, white wrt t)

$$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt} \right) = \nabla \cdot u^S = 0$$

Newton 2nd law

Momentum conservation
$$\text{“}d(dX_t) = \text{Forces}(t, X_t, u_t)\text{”}$$

Positions of fluid parcels X_t :
$$dX_t = u_t^S(X_t)dt + \sigma_t(X_t) \circ dB_t$$

?

?

Mikulevicius & Rozovskii (2004)

Momentum conservation
$$d\left(u_t^S(X_t)\right) = F(t, X_t, u_t)dt + F_\sigma(t, X_t, u_t) \circ dB_t \text{ (Forces)}$$

Positions of fluid parcels X_t :
$$dX_t = u_t^S(X_t)dt + \sigma_t(X_t) \circ dB_t$$

Mémin (2014)

Momentum conservation
$$d\left(u_t^I(X_t)\right) = F(t, X_t, u_t)dt + F_\sigma(t, X_t, u_t) \circ dB_t \text{ (Forces)}$$

Positions of fluid parcels X_t :
$$dX_t = u_t^I(X_t)dt + \sigma_t(X_t)dB_t$$

Transport noise
Stochastic Navier-Stokes equations under location uncertainty (LU)

$v = u + v'$

Resolved fluid velocity:
u

Unresolved fluid velocity:
$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt}$
$:= \sigma(x) \circ \frac{dB_t}{dt}$
increments of **Q-Wiener** process
(Gaussian, white wrt t)

$$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt} \right) = \nabla \cdot u = 0$$

Momentum conservation

$$d\big(u_t(X_t)\big) = F(t, X_t, u_t)dt + \underbrace{F_\sigma(t, X_t, u_t) \circ dB_t}_{\text{conditionnally Gaussian process, white in time}} \text{ (Forces)}$$

conditionnally
Gaussian
process
white in time

Positions of fluid parcels X_t :

$$dX_t = u_t(X_t)dt + \underbrace{\sigma_t(X_t) \circ dB_t}_{\text{conditionnally Gaussian process, white in time}}$$

conditionnally
Gaussian
process
white in time

Resseguier & al. (2017, 2020, 2021)

Transport noise
Stochastic Navier-Stokes equations under location uncertainty (LU)

$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:
$$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt}$$
$$:= \sigma(x) \circ \frac{dB_t}{dt}$$
increments of **Q-Wiener** process
(Gaussian, white wrt t)

$$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt} \right) = \nabla \cdot u = 0$$

Momentum conservation

$$d\big(u_t(X_t)\big) = dF_t \text{ (Forces)}$$

Positions of fluid parcels X_t :

$$dX_t = u_t(X_t)dt + \underbrace{\sigma_t(X_t) \circ dB_t}_{}$$

conditionnally
Gaussian
process
white in time

Resseguier & al. (2017, 2020, 2021)

Transport noise
Stochastic Navier-Stokes equations under location uncertainty (LU)

From Ito-Wentzell formula (Kunita 1990) with Stratonovich notations

$v = u + v'$

Resolved fluid velocity:
u

Unresolved fluid velocity:
$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt}$
$:= \sigma(x) \circ \frac{dB_t}{dt}$
increments of **Q-Wiener** process
(Gaussian, white wrt t)

$$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt} \right) = \nabla \cdot u = 0$$

$$d_t u_t + u_t \cdot \nabla u_t dt + \sigma \circ dB_t \cdot \nabla u_t = dF_t$$

Advection | Forces

Kunita (1990)
Resseguier & al. (2017, 2020, 2021)

Transport noise
Stochastic Navier-Stokes equations under location uncertainty (LU)

From Ito-Wentzell formula (Kunita 1990) with Stratonovich notations

$v = u + v'$

Resolved fluid velocity: u

Unresolved fluid velocity:

$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt}$

$:= \sigma(x) \circ \frac{dB_t}{dt}$

increments of **Q-Wiener** process (Gaussian, white wrt t)

$$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt} \right) = \nabla \cdot u = 0$$

Usual terms

$$d_t u_t + u_t \cdot \nabla u_t dt + \sigma \circ dB_t \cdot \nabla u_t = dF_t$$

Advection

Forces

Kunita (1990)
Resseguier & al. (2017, 2020, 2021)

Transport noise
Stochastic Navier-Stokes equations under location uncertainty (LU)

From Ito-Wentzell formula (Kunita 1990) with Stratonovich notations

$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt} := \sigma(x) \circ \frac{dB_t}{dt}$$

increments of **Q-Wiener** process (Gaussian, white wrt t)

$$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt} \right) = \nabla \cdot u = 0$$

Usual terms

$$d_t u_t + u_t \cdot \nabla u_t dt + \sigma \circ dB_t \cdot \nabla u_t = dF_t$$

Advection

Forces

Skew-symmetric multiplicative random forcing

Kunita (1990)
Resseguier & al. (2017, 2020, 2021)

Transport noise

Stochastic Navier-Stokes equations under location uncertainty (LU)

From Ito-Wentzell formula (Kunita 1990) with Stratonovich notations

$v = u + v'$

Resolved fluid velocity: u

Unresolved fluid velocity:

$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt}$

$:= \sigma(x) \circ \frac{dB_t}{dt}$

increments of **Q-Wiener** process (Gaussian, white wrt t)

$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt} \right) = \nabla \cdot u = 0$

Usual terms

$$d_t u_t + u_t \cdot \nabla u_t dt + \sigma \circ dB_t \cdot \nabla u_t = dF_t$$

Advection

Forces

Skew-symmetric multiplicative random forcing

Energy conservation

$$\frac{d}{dt} \int \|u_t(x)\|^2 dx = 0$$

Kunita (1990)
Resseguier & al. (2017, 2020, 2021)

Transport noise
Stochastic Navier-Stokes equations under location uncertainty (LU)

From Ito-Wentzell formula (Kunita 1990) with Ito notations

$$v = u + v'$$

Resolved fluid velocity: u

Unresolved fluid velocity:

$$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt} := \sigma(x) \circ \frac{dB_t}{dt}$$

increments of **Q-Wiener** process (Gaussian, white wrt t)

$$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt} \right) = \nabla \cdot u = 0$$

Usual terms

$$d_t u_t + u_t \cdot \nabla u_t dt + \sigma dB_t \cdot \nabla u_t = \nabla \cdot (A \nabla u_t) + dF_t$$

Advection | Diffusion | Forces

Skew-symmetric multiplicative random forcing

Kunita (1990)
Resseguier & al. (2017, 2020, 2021)

Transport noise
Stochastic Navier-Stokes equations under location uncertainty (LU)

From Ito-Wentzell formula (Kunita 1990) with Ito notations

$v = u + v'$

Resolved fluid velocity: u

Unresolved fluid velocity:

$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt}$

$:= \sigma(x) \circ \frac{dB_t}{dt}$

increments of **Q-Wiener** process (Gaussian, white wrt t)

$$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt} \right) = \nabla \cdot u = 0$$

Usual terms

$$d_t u_t + u_t \cdot \nabla u_t dt + \sigma dB_t \cdot \nabla u_t = \nabla \cdot (A \nabla u_t) + dF_t$$

Advection — Diffusion — Forces

Skew-symmetric multiplicative random forcing

Energy conservation

$$\frac{d}{dt} \int \|u_t(x)\|^2 dx = 0$$

Kunita (1990)
Resseguier & al. (2017, 2020, 2021)

Transport noise
Stochastic Navier-Stokes equations under location uncertainty (LU)

$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:
$$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt}$$
$$:= \sigma(x) \circ \frac{dB_t}{dt}$$
increments of **Q-Wiener** process
(Gaussian, white wrt t)

$$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt} \right) = \nabla \cdot u = 0$$

Usual terms

$$d_t u_t + C(u_t, u_t)dt + C(\sigma \circ dB_t, u_t) = dF_t$$

Advection

Forces

Skew-symmetric
multiplicative
random
forcing

Broader scope: SPDEs with transport noise

SALT, LU, …

$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:
$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt}$
$:= \sigma(x) \circ \frac{dB_t}{dt}$
increments of **Q-Wiener** process
(Gaussian, white wrt t)

$$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt}\right) = \nabla \cdot u = 0$$

Multiplicative noise
(transport noise)

$$d_t q_t + C(u_t, q_t)dt + C(\sigma \circ dB_t, q_t) = dS_t^q$$

Advection

Resseguier & al. (2021), Zhen et al. (2023)

Broader scope: SPDEs with transport noise

SALT, LU, …

$v = u + v'$

Resolved fluid velocity: u

Unresolved fluid velocity:

$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt}$

$:= \sigma(x) \circ \frac{dB_t}{dt}$

increments of **Q-Wiener** process (Gaussian, white wrt t)

$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt} \right) = \nabla \cdot u = 0$

Multiplicative noise (transport noise)

$$d_t q_t + C(u_t, q_t)dt + C(\sigma \circ dB_t, q_t) = dS_t^q$$

Advection

Examples :

- q could be
  - T (0-form)
  - ρ (n-form)
  - u (1-form)
  - …
- $C(v,q)$ could be
  - $(v \cdot \nabla)q$
  - $\nabla \cdot (vq)$
  - $(v \cdot \nabla)q + \nabla v^T q$
  - $\mathcal{L}_v q$
  - …

Resseguier & al. (2021), Zhen et al. (2023)

Broader scope: SDEs with transport noise

SALT, LU, …

Multiplicative noise
(transport noise)

$v = u + v'$

Resolved fluid velocity:
u

Unresolved fluid velocity:
$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt}$
$:= \sigma(x) \circ \frac{dB_t}{dt}$
increments of **Q-Wiener** process
(Gaussian, white wrt t)

$$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt}\right) = \nabla \cdot u = 0$$

Spatial discretization
$(Q_t)_i = q_t(x_i)$
$G_{ik}(Q_t) = \big(C(\xi_k, q_t)\big)(x_i)$

$$dQ_t = F(t, Q_t)dt + G(Q_t) \circ d\tilde{B}_t + \cdots$$

Examples :

- q could be
  - T (0-form)
  - ρ (n-form)
  - u (1-form)
  - …
- $C(v,q)$ could be
  - $(v \cdot \nabla)q$
  - $\nabla \cdot (vq)$
  - $(v \cdot \nabla)q + \nabla v^T q$
  - $\mathcal{L}_v q$
  - …

Resseguier & al. (2021), Zhen et al. (2023)

Transport noise

Conclusion

$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt}$$

$$:= \sigma(x) \circ \frac{dB_t}{dt}$$

increments of **Q-Wiener** process
(Gaussian, white wrt t)

Assumed
(conditionally-)Gaussian
& white in time
(non-stationary in space)

$$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt} \right) = \nabla \cdot u = 0$$

Transport noise
Conclusion

$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt}$$
$$:= \sigma(x) \circ \frac{dB_t}{dt}$$

increments of **Q-Wiener** process
(Gaussian, white wrt t)

Assumed
(conditionally-)Gaussian
& white in time
(non-stationary in space)

$$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt} \right) = \nabla \cdot u = 0$$

Stochastic Navier-Stokes models and other SPDEs

- Closure for coarse-grid simulations
- Quantification of numerical model error **for data assimilation / filtering**
- Many theoretical problems (wellposedness, …)

Transport noise
Conclusion

$v = u + v'$

Resolved fluid velocity: u

Unresolved fluid velocity:

$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt}$

$:= \sigma(x) \circ \frac{dB_t}{dt}$

increments of **Q-Wiener** process (Gaussian, white wrt t)

$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt} \right) = \nabla \cdot u = 0$

Assumed (conditionally-)Gaussian **& white in time** (non-stationary in space)

Stochastic Navier-Stokes models and other SPDEs

- Closure for coarse-grid simulations
- Quantification of numerical model error **for data assimilation / filtering**
- Many theoretical problems (wellposedness, …)

References : Kraichnan, 1968

Mikulevicius & Rozovskii, 2004
Flandoli, 2011

LU	SALT	
Memin, 2014 Resseguier et al. 2017 a, b, c, d Cai et al. 2017 Chapron et al. 2018 Yang & Memin 2019 …	Holm, 2015 Holm and Tyranowski, 2016 Arnaudon et al. 2017	Crisan et al., 2017 Gay-Balmaz & Holm 2017 Cotter and al. 2018 a, b Cotter and al. 2019 …

Cotter and al. 2017, Resseguier et al. 2020, 2021, Zhen et al. (2023), …

Transport noise
Conclusion

$$v = u + v'$$

Resolved fluid velocity: u

Unresolved fluid velocity:

$$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt}$$
$$:= \sigma(x) \circ \frac{dB_t}{dt}$$

increments of **Q-Wiener** process (Gaussian, white wrt t)

Assumed (conditionally-)Gaussian **& white in time** (non-stationary in space)

$$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt} \right) = \nabla \cdot u = 0$$

Stochastic Navier-Stokes models and other SPDEs

- Closure for coarse-grid simulations
- Quantification of numerical model error **for data assimilation / filtering**
- Many theoretical problems (wellposedness, …)

References : Kraichnan, 1968

Mikulevicius & Rozovskii, 2004
Flandoli, 2011

LU
Memin, 2014
Resseguier et al. 2017 a, b, c, d
Cai et al. 2017
Chapron et al. 2018
Yang & Memin 2019
…

SALT
Holm, 2015
Holm and Tyranowski, 2016
Arnaudon et al. 2017

Crisan et al., 2017
Gay-Balmaz & Holm 2017
Cotter and al. 2018 a, b
Cotter and al. 2019
…

Cotter and al. 2017, Resseguier et al. 2020, 2021, Zhen et al. (2023), …

erc STUOD
2020 - 2026
+ Many papers !!!
from teams of PIs:
- B Chapron
- D Crisan
- D D Holm
- E Mémin

Transport noise

Conclusion

$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sum_i \xi_i(x) \circ \frac{dW_t^i}{dt}$$
$$:= \sigma(x) \circ \frac{dB_t}{dt}$$

increments of **Q-Wiener** process
(Gaussian, white wrt t)

Assumed
(conditionally-)Gaussian
& white in time
(non-stationary in space)

$$\nabla \cdot \left(\sigma \circ \frac{dB_t}{dt} \right) = \nabla \cdot u = 0$$

Stochastic Navier-Stokes models and other SPDEs

- Closure for coarse-grid simulations
- Quantification of numerical model error **for data assimilation / filtering**
- Many theoretical problems (wellposedness, …)

Lang & Crisan (2023, 2024)
Goodair et al. (2024)

References :

Kraichnan, 1968

Mikulevicius & Rozovskii, 2004
Flandoli, 2011

LU	SALT	
Memin, 2014 Resseguier et al. 2017 a, b, c, d Cai et al. 2017 Chapron et al. 2018 Yang & Memin 2019 …	Holm, 2015 Holm and Tyranowski, 2016 Arnaudon et al. 2017	Crisan et al., 2017 Gay-Balmaz & Holm 2017 Cotter and al. 2018 a, b Cotter and al. 2019 …

Cotter and al. 2017, Resseguier et al. 2020, 2021, Zhen et al. (2023), …

2020 - 2026
+ Many papers !!!
from teams of PIs:

- B Chapron
- D Crisan
- D D Holm
- E Mémin

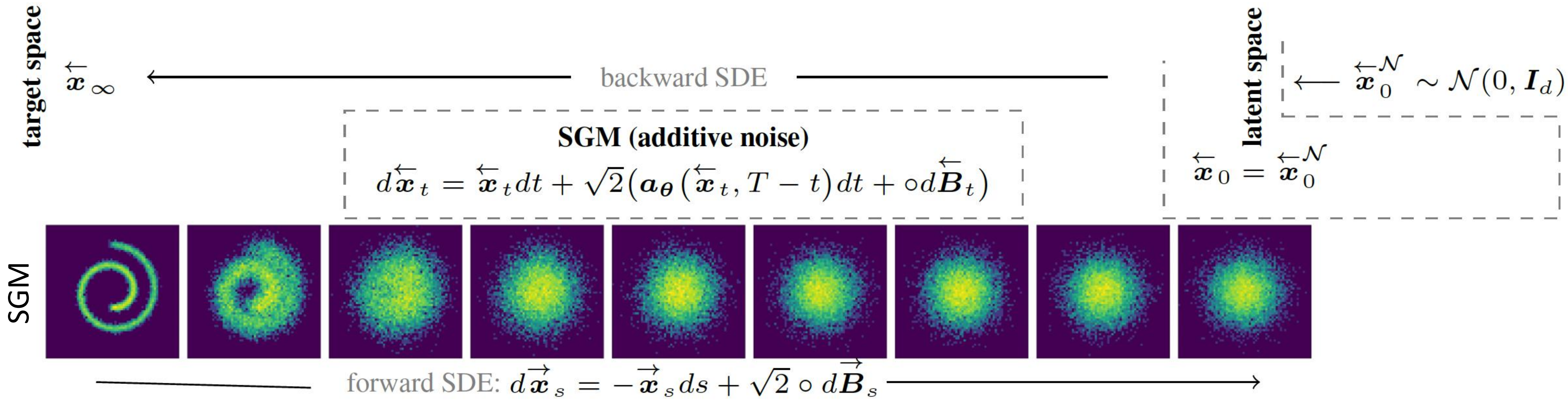
III. Multiplicative score-based generative model

- Big picture
- Dynamics on d-spheres & Fokker-Planck equations
- Latent space
- Backward diffusion with a neural network & ELBO
- Numerical results
- Open questions

In collaboration with : Merveille Talla, Robert Gruhlke, Dominique Heitz , Etienne Mémin

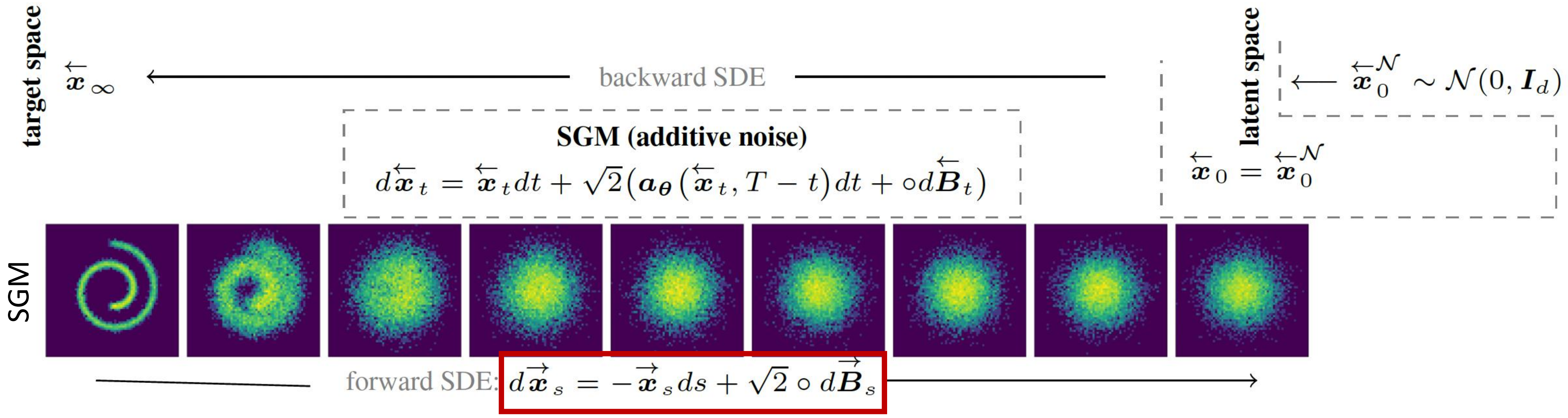



State-of-the-art : Diffusion models / DDPM / SGM



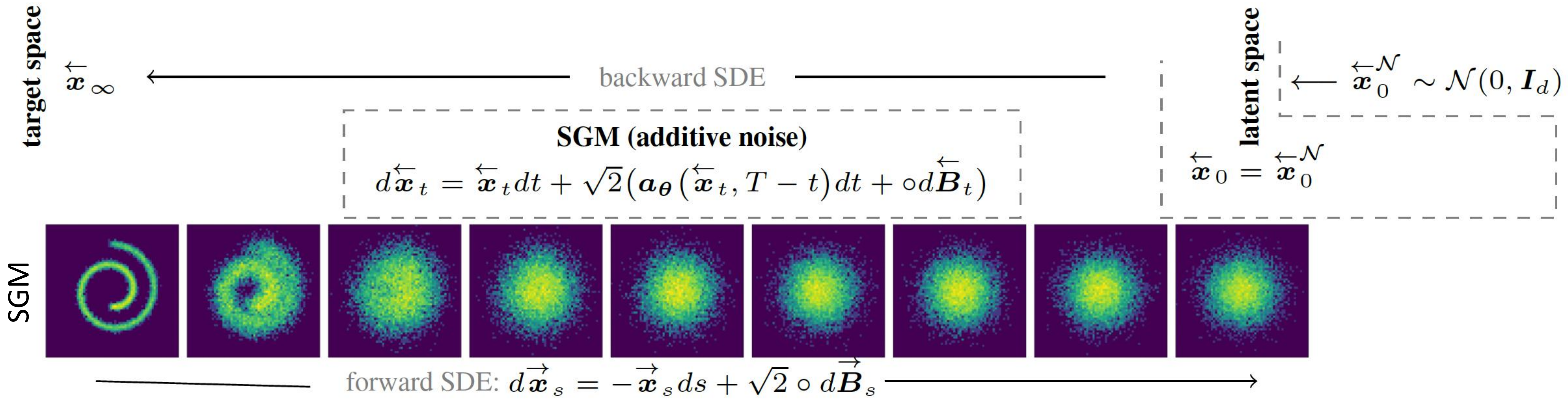
Song et al. (2021). in *ICLR*

State-of-the-art : Diffusion models / DDPM / SGM



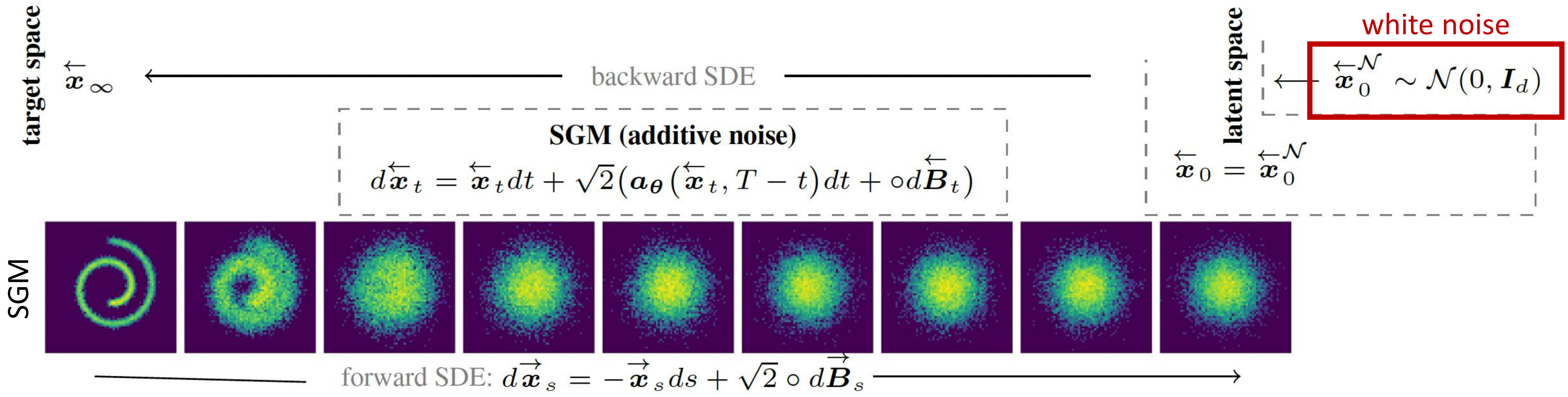
Song et al. (2021). in *ICLR*

State-of-the-art : Diffusion models / DDPM / SGM



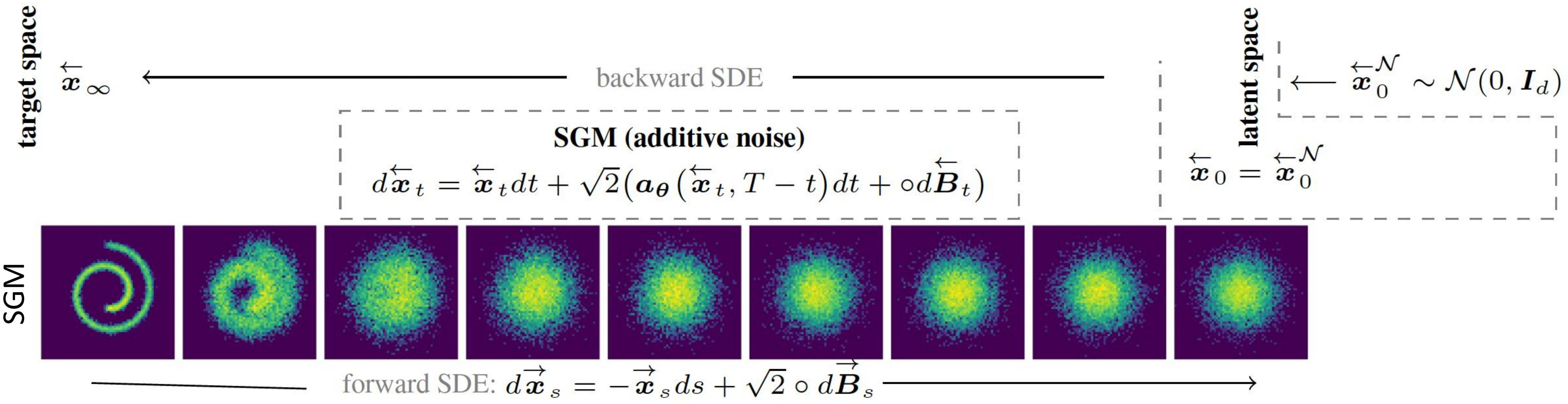
Song et al. (2021). in *ICLR*

State-of-the-art : Diffusion models / DDPM / SGM



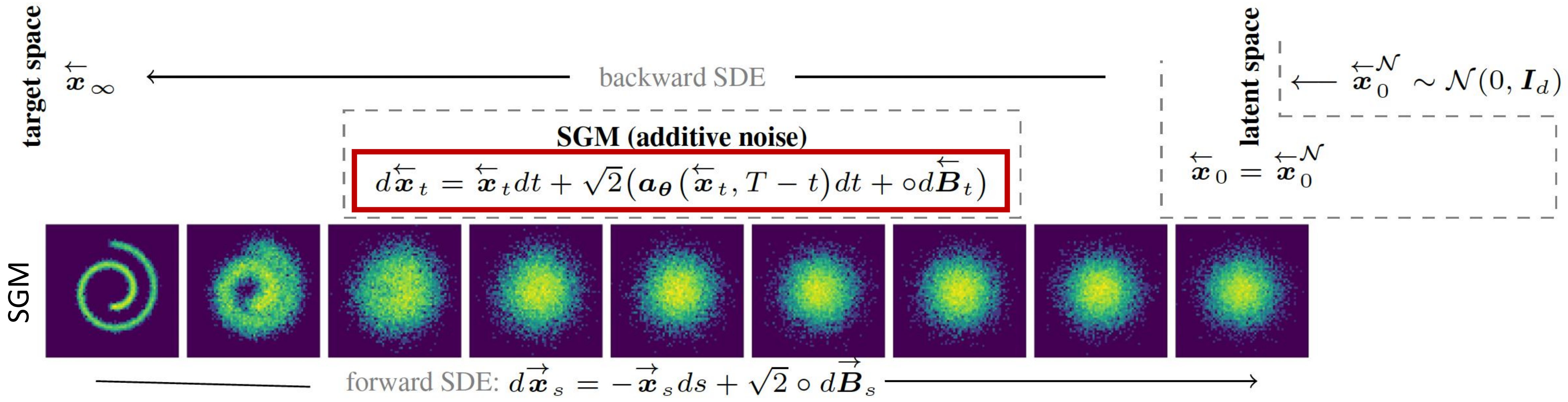
Song et al. (2021). in *ICLR*

State-of-the-art : Diffusion models / DDPM / SGM



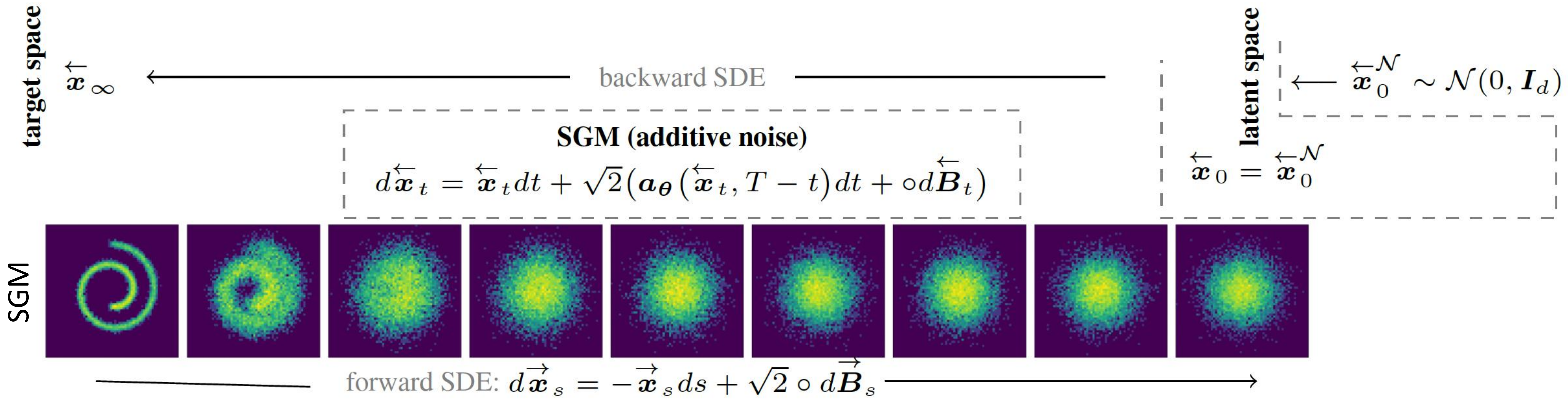
Song et al. (2021). in *ICLR*

State-of-the-art : Diffusion models / DDPM / SGM



Song et al. (2021). in *ICLR*

State-of-the-art : Diffusion models / DDPM / SGM

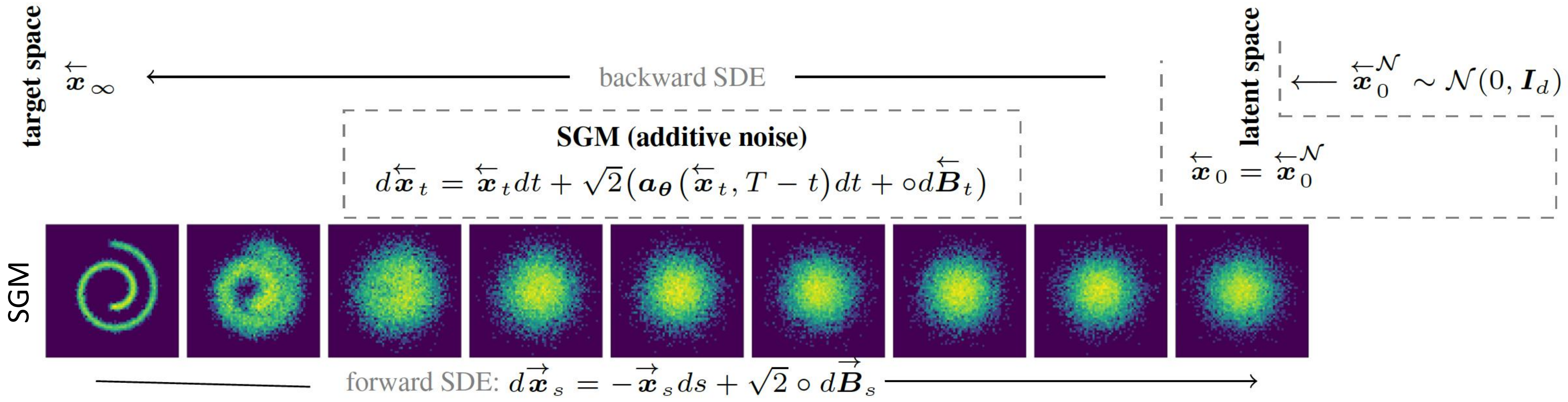


Song et al. (2021). in *ICLR*

State-of-the-art : Diffusion models / DDPM / SGM

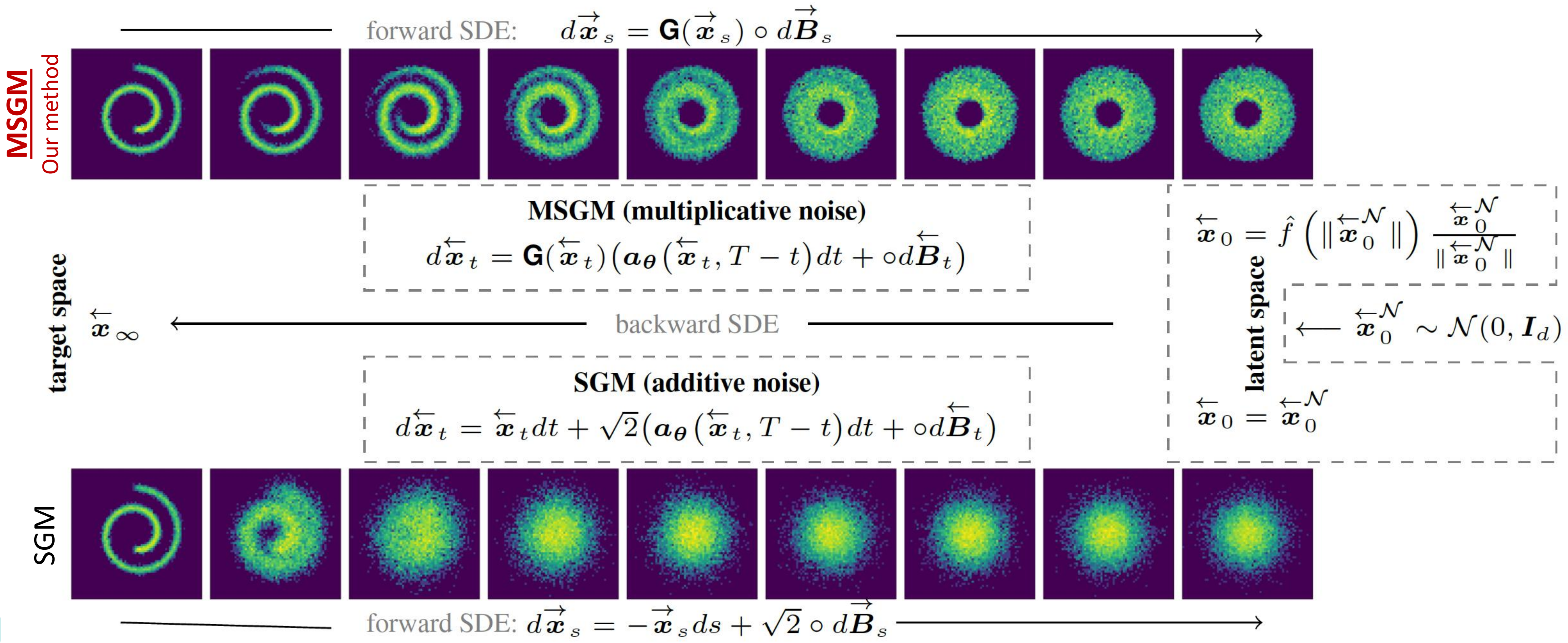
Limitations of diffusion models / DDPM / SGM

- Hardly represent rare events
- Need a lot of data
- Computational cost for reverse process
- For applications in Physics, no underlying physical structure



Song et al. (2021). in *ICLR*

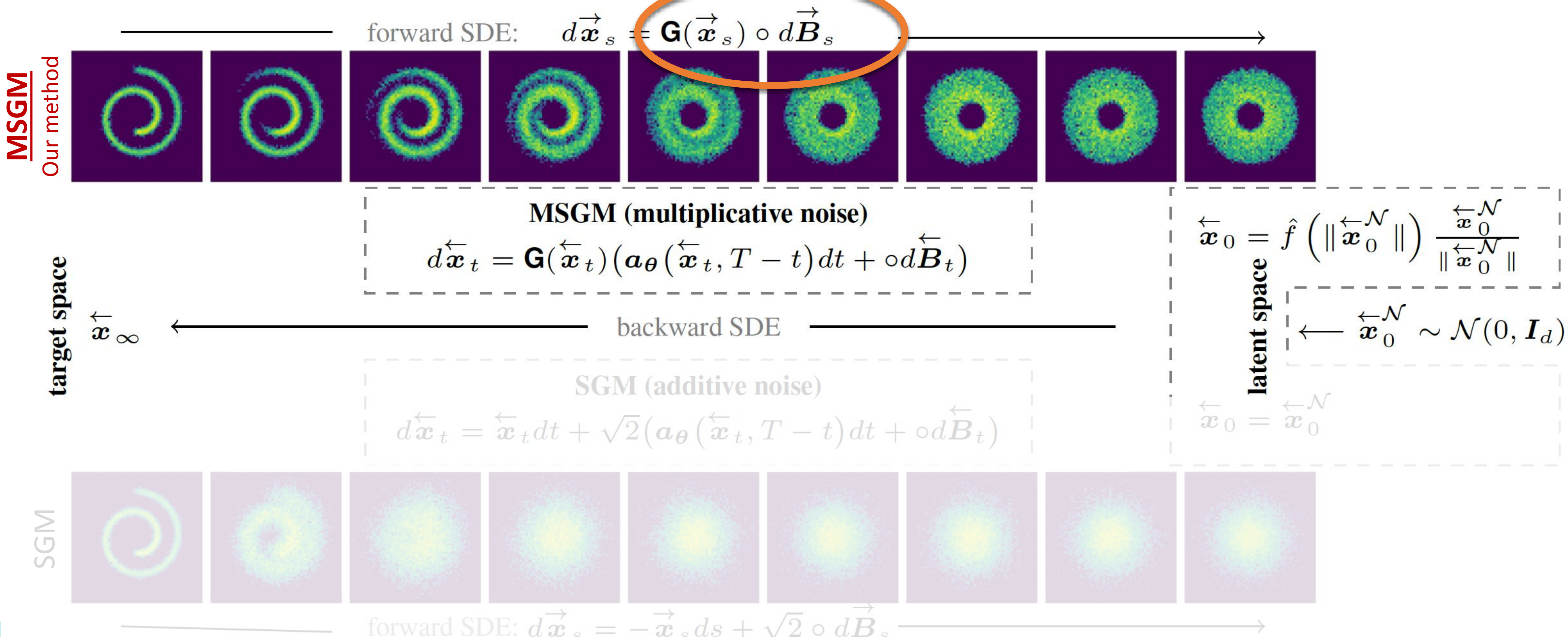
A new generative method : multiplicative diffusion models (MSGM)



Gruhlke et al. (2026) in *ICLR*

A new generative method : multiplicative diffusion models (MSGM)

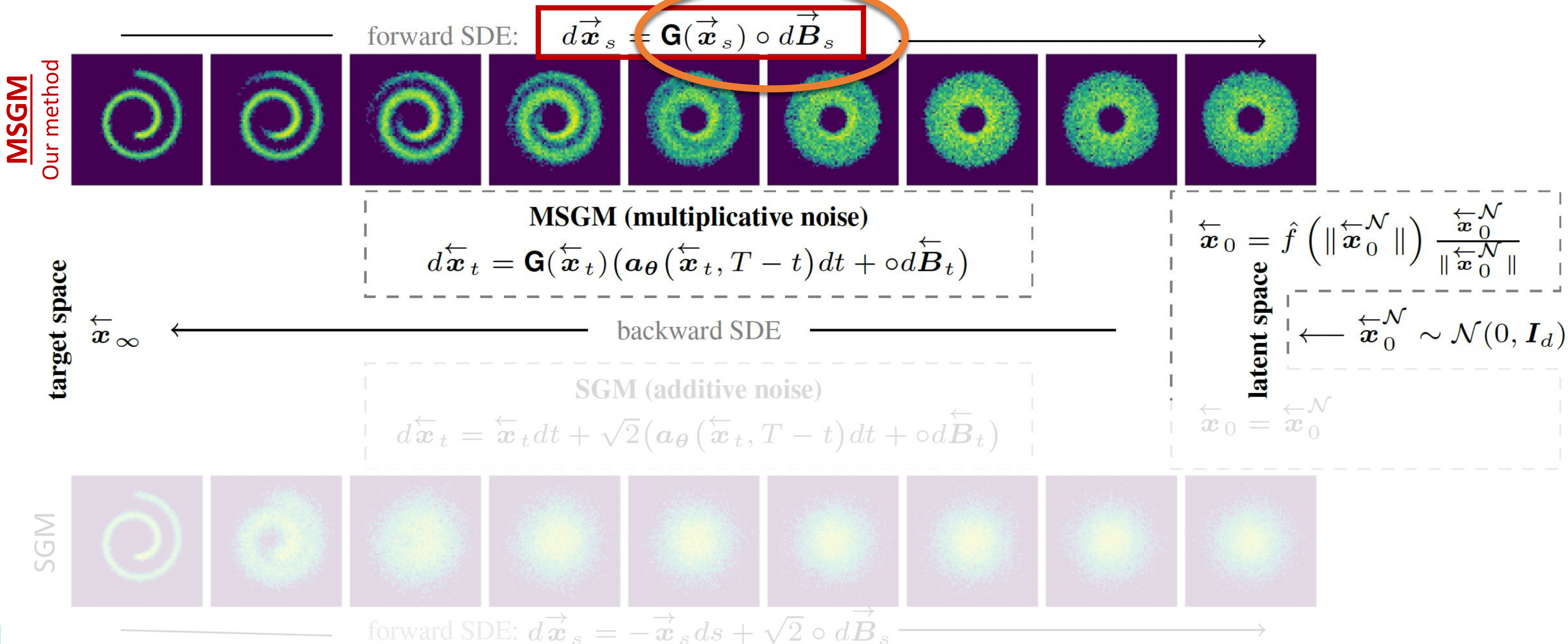
Multiplicative noise (e.g. transport noise)



Gruhlke et al. (2026) in *ICLR*

A new generative method : multiplicative diffusion models (MSGM)

Multiplicative noise (e.g. transport noise)



Gruhlke et al. (2026) in *ICLR*

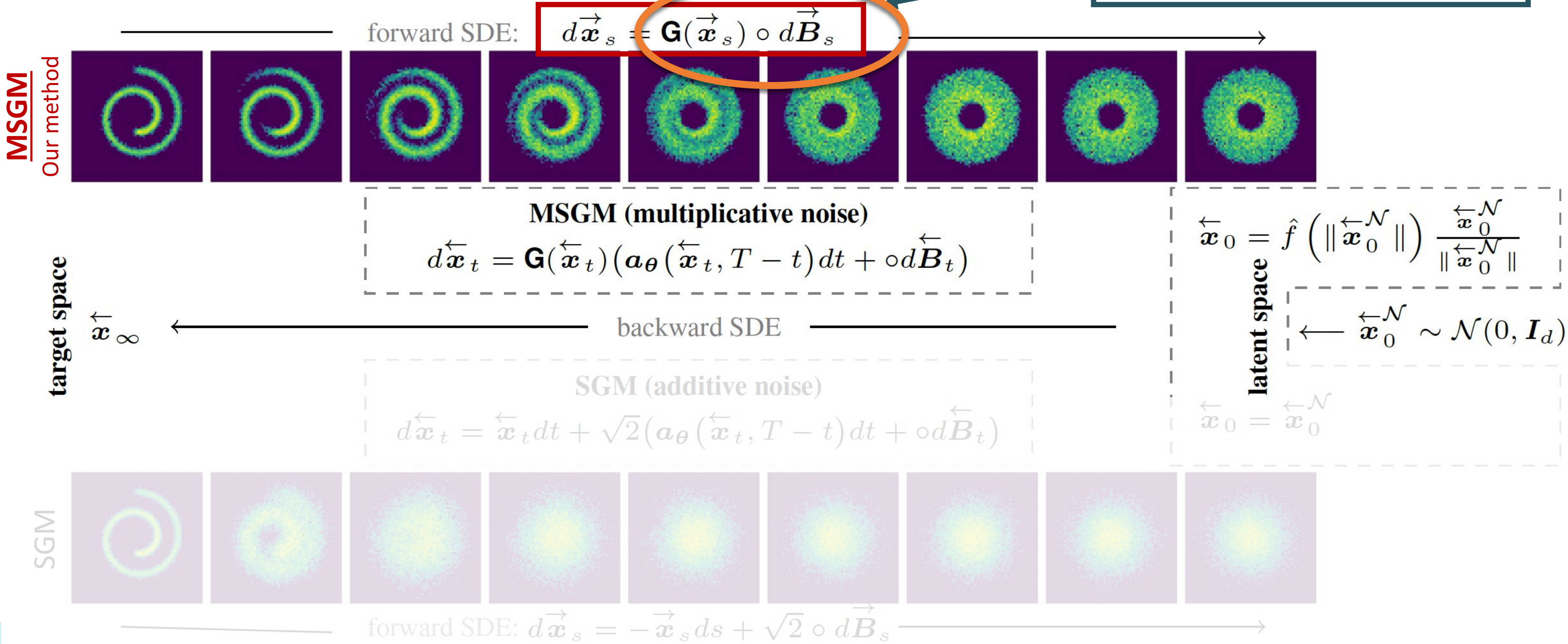
A new generative method : multiplicative diffusion models (MSGM)

Random rotations

$$x_{s+ds} = x_{s-ds} + \underbrace{Z_{ds}} \, x_s$$

- skew-symmetric matrix
- white noise in time

Multiplicative noise (e.g. transport noise)



Gruhlke et al. (2026) in *ICLR*

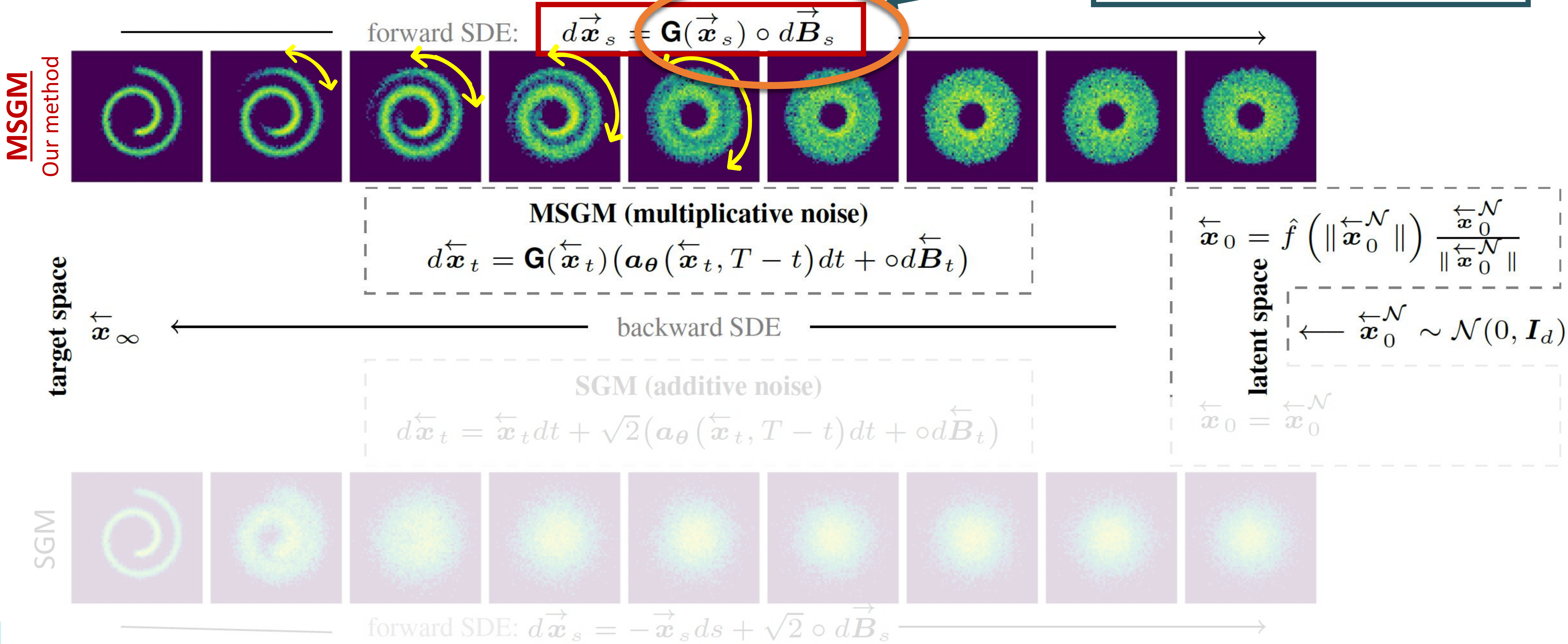
A new generative method : multiplicative diffusion models (MSGM)

Random rotations

$$x_{s+ds} = x_{s-ds} + \underbrace{Z_{ds}}\ x_s$$

- skew-symmetric matrix
- white noise in time

Multiplicative noise (e.g. transport noise)



Gruhlke et al. (2026) in *ICLR*

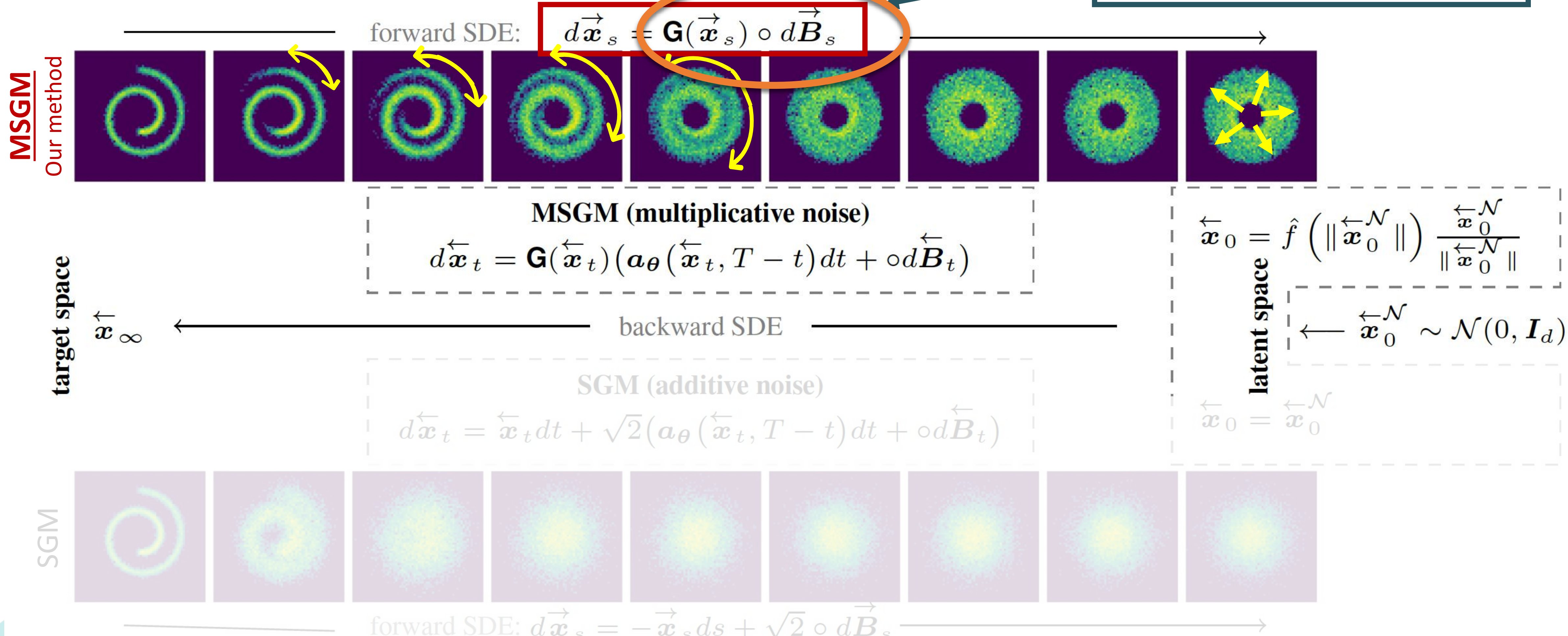
A new generative method : multiplicative diffusion models (MSGM)

Random rotations

$$x_{s+ds} = x_{s-ds} + \underbrace{Z_{ds}}\ x_s$$

- skew-symmetric matrix
- white noise in time

Multiplicative noise (e.g. transport noise)



Gruhlke et al. (2026) in *ICLR*

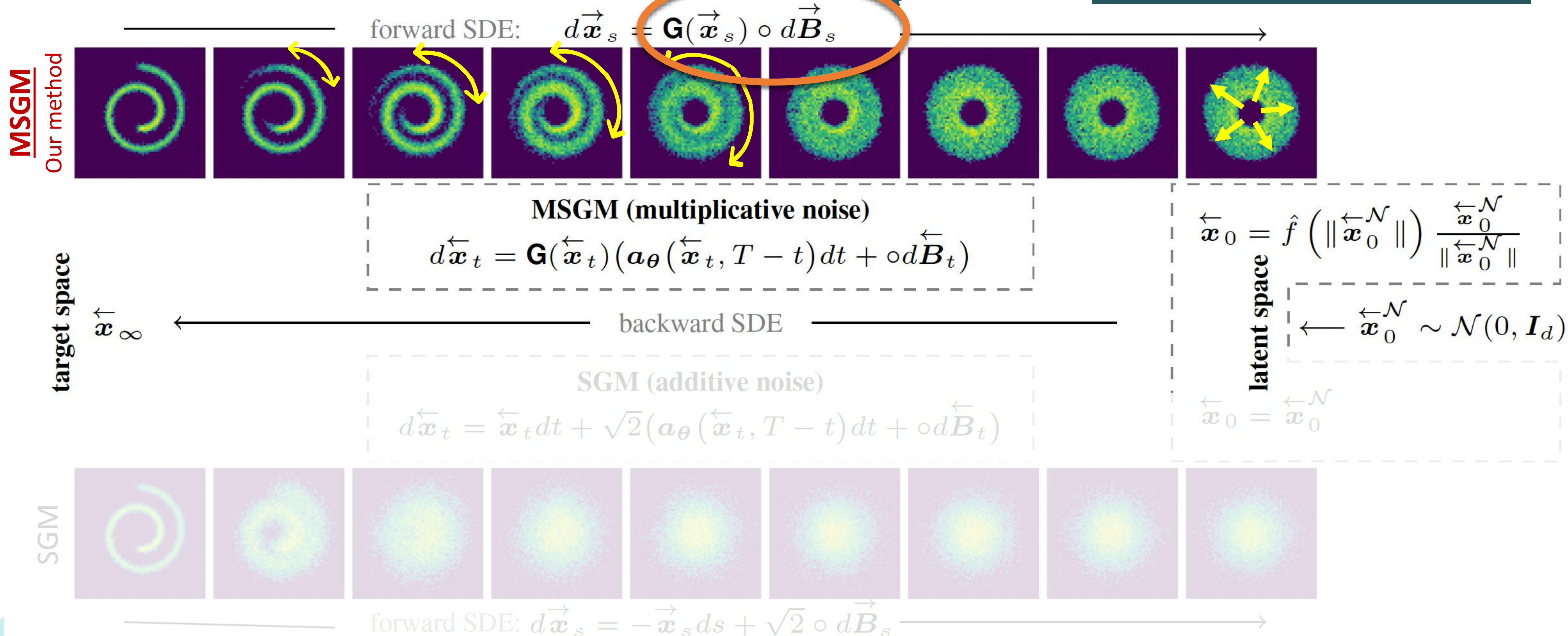
A new generative method : multiplicative diffusion models (MSGM)

Random rotations

$$x_{s+ds} = x_{s-ds} + \underbrace{\mathbf{Z}_{ds}}\ x_s$$

- skew-symmetric matrix
- white noise in time

Multiplicative noise (e.g. transport noise)



Gruhlke et al. (2026) in *ICLR*

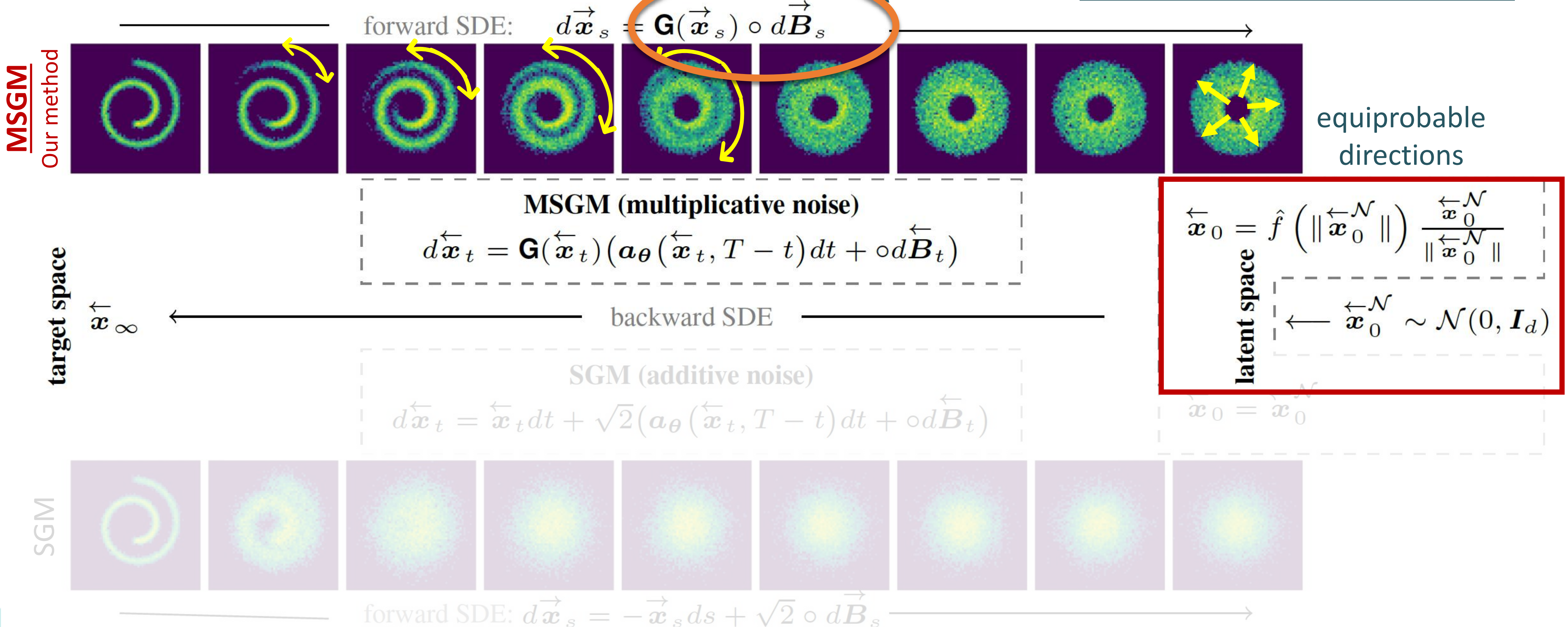
A new generative method : multiplicative diffusion models (MSGM)

Random rotations

$$x_{s+ds} = x_{s-ds} + \underbrace{Z_{ds}}\ x_s$$

- skew-symmetric matrix
- white noise in time

Multiplicative noise (e.g. transport noise)



Gruhlke et al. (2026) in *ICLR*

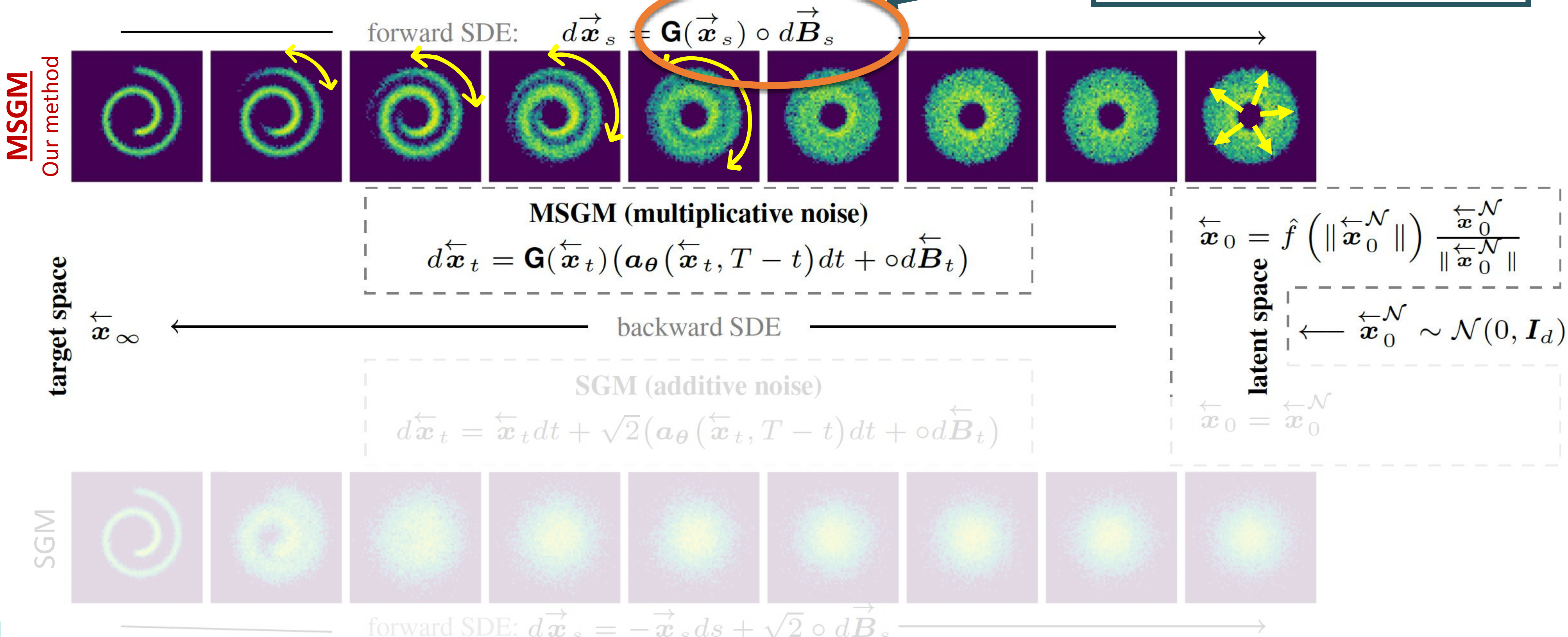
A new generative method : multiplicative diffusion models (MSGM)

Random rotations

$$x_{s+ds} = x_{s-ds} + \underbrace{Z_{ds}} \; x_s$$

- skew-symmetric matrix
- white noise in time

Multiplicative noise (e.g. transport noise)



Gruhlke et al. (2026) in *ICLR*

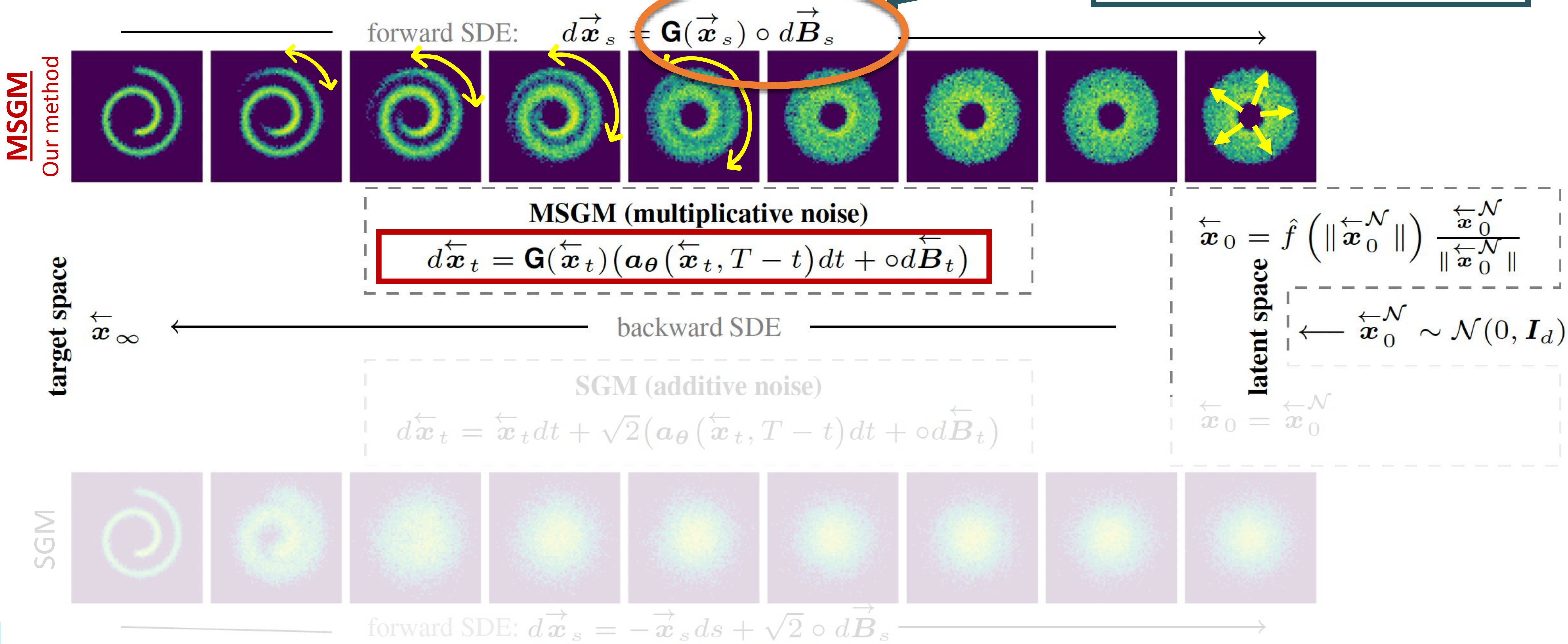
A new generative method : multiplicative diffusion models (MSGM)

Random rotations

$$x_{s+ds} = x_{s-ds} + \underbrace{Z_{ds}} \; x_s$$

- skew-symmetric matrix
- white noise in time

Multiplicative noise (e.g. transport noise)



Gruhlke et al. (2026) in *ICLR*

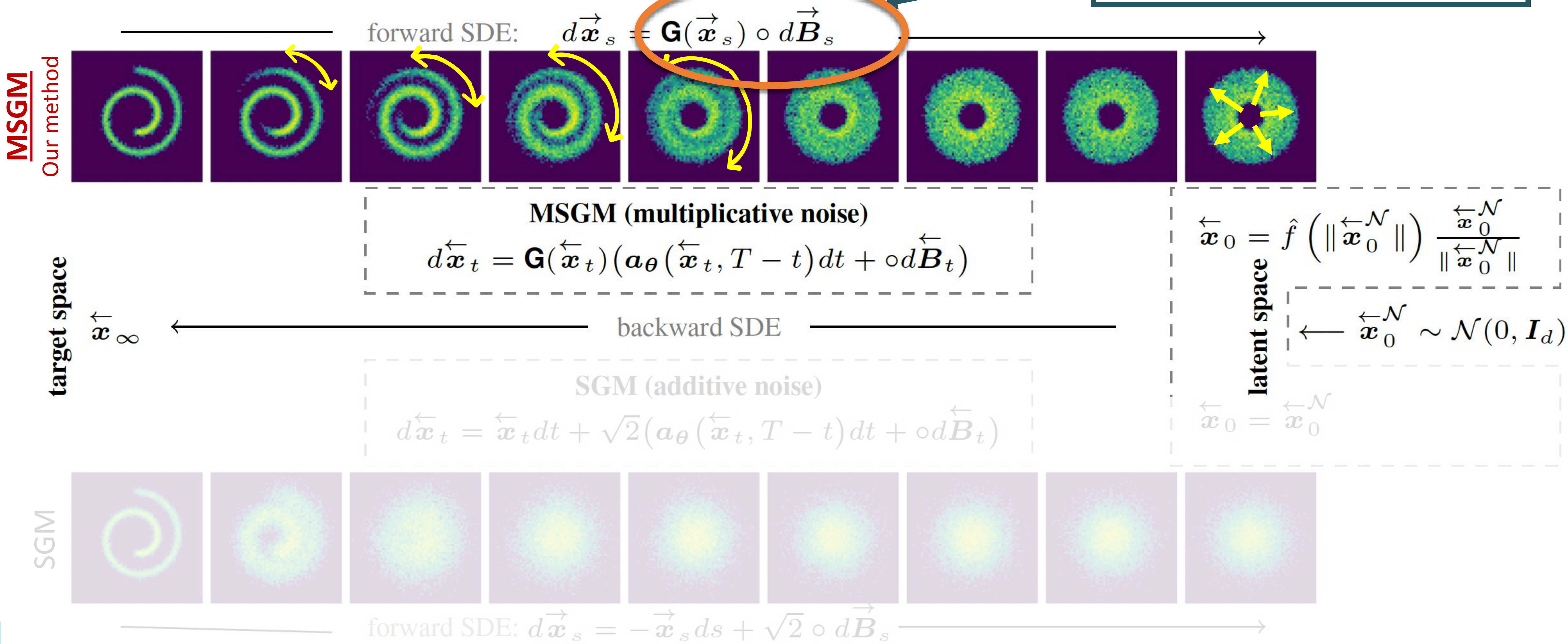
A new generative method : multiplicative diffusion models (MSGM)

Random rotations

$$x_{s+ds} = x_{s-ds} + \underbrace{Z_{ds}}\ x_s$$

- skew-symmetric matrix
- white noise in time

Multiplicative noise (e.g. transport noise)



Gruhlke et al. (2026) in *ICLR*

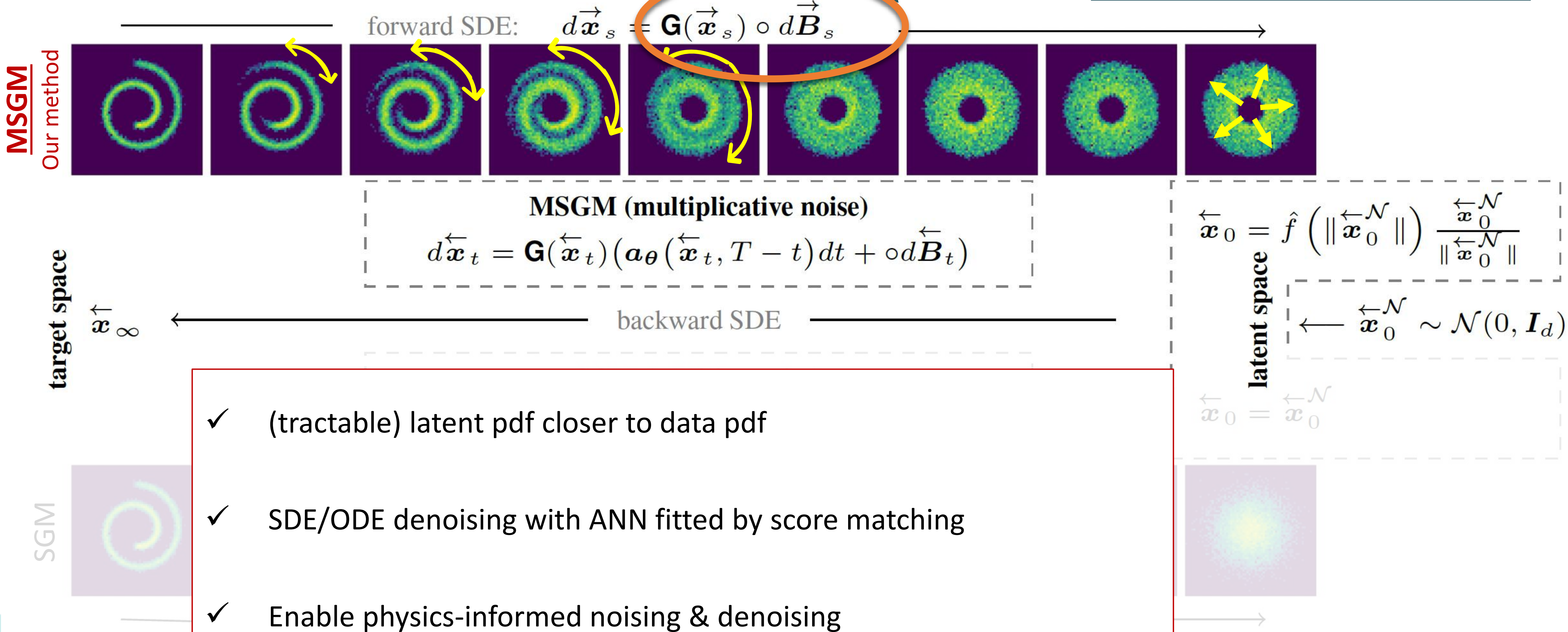
A new generative method : multiplicative diffusion models (MSGM)

Multiplicative noise (e.g. transport noise)

Random rotations

$$x_{s+ds} = x_{s-ds} + \underbrace{\mathbf{Z}_{ds}}\ x_s$$

- skew-symmetric matrix
- white noise in time



- ✓ (tractable) latent pdf closer to data pdf
- ✓ SDE/ODE denoising with ANN fitted by score matching
- ✓ Enable physics-informed noising & denoising

Gruhlke et al. (2026) in *ICLR*

Forward diffusion

$d\vec{x}_s = G(\vec{x}_s) \circ d\vec{B}_t$ Or $\boldsymbol{d\vec{x}_s = [\circ\, d\vec{Z}_s]\vec{x}_s}$

with $G(x) = [G^1x \ \dots \ G^dx]$ **with** $\boldsymbol{\vec{Z}_s = \sum_{k=1}^{d} G^k \left(\vec{B}_t\right)_k}$

- **Assumptions**

1) $\left(\mathrm{G}^{\mathrm{k}}\right)_{\mathrm{k}}$ skew-symmetric matrices

2) $\mathrm{Im}\big(G(x)\big) = x^{\perp}, \ \ \forall x \neq 0$

Gruhlke et al. (2026) in *ICLR*

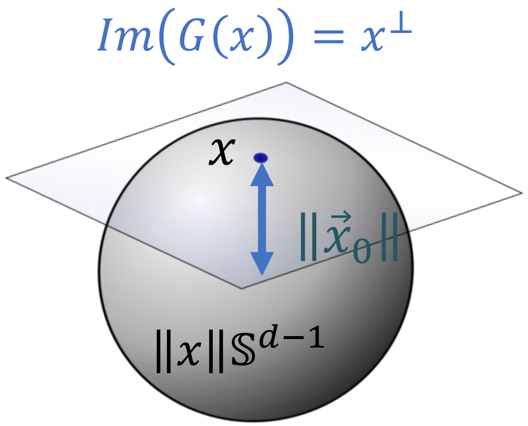
Forward diffusion

$d\vec{x}_s = G(\vec{x}_s) \circ d\vec{B}_t$

with $G(x) = [G^1 x \ \dots \ G^d x]$

Or $\boldsymbol{d\vec{x}_s = [\circ\, d\vec{Z}_s]\vec{x}_s}$

with $\boldsymbol{\vec{Z}_s = \sum_{k=1}^{d} G^k \left(\vec{B}_t\right)_k}$



- **Assumptions**

1) $(\mathrm{G}^{\mathrm{k}})_{\mathrm{k}}$ skew-symmetric matrices

2) $\mathrm{Im}(G(x)) = x^{\perp}, \ \ \forall x \neq 0$

Gruhlke et al. (2026) in *ICLR*

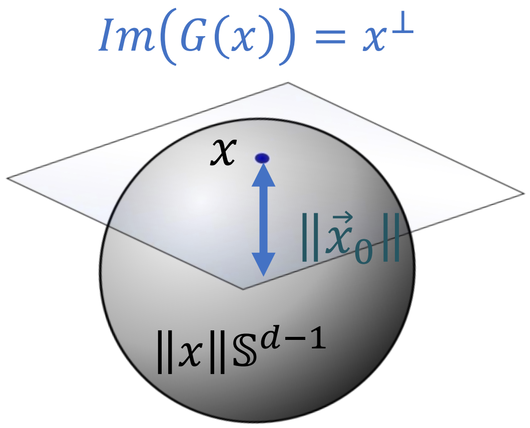
Forward diffusion

$d\vec{x}_s = G(\vec{x}_s) \circ d\vec{B}_t$

with $G(x) = [G^1 x \ \dots \ G^d x]$

Or $\boldsymbol{d\vec{x}_s = [\circ \, d\vec{Z}_s]\vec{x}_s}$

with $\boldsymbol{\vec{Z}_s = \sum_{k=1}^{d} G^k \left(\vec{B}_t\right)_k}$



- **Assumptions**

1) $(\mathrm{G^k})_\mathrm{k}$ skew-symmetric matrices

2) $\mathrm{Im}(G(x)) = x^{\perp}, \ \ \forall x \neq 0$

What happens to the distribution of $\vec{x}_t$ when $t \ \rightarrow \ \infty$?

Gruhlke et al. (2026) in *ICLR*

Fokker-Planck equation

Theorem 1 (Gruhlke, Resseguier, Talla. 2026)

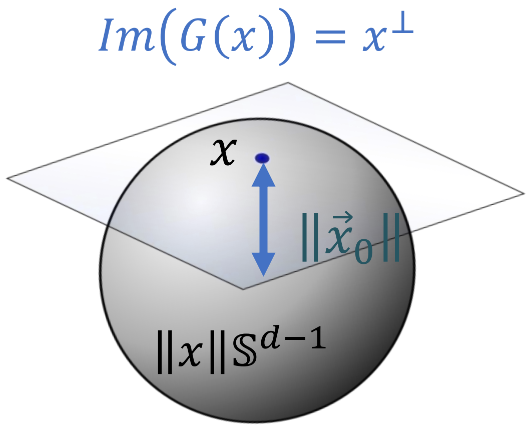
Under 1),

$$\frac{\partial p_s}{\partial s}(x) = \nabla_\perp \cdot \left(\frac{1}{2}\Sigma(x)\nabla_\perp p_s(x)\right), x \in \mathbb{R}^d.$$

with $\Sigma(x) := G(x)G(x)^\intercal$, $x^n = \frac{x}{\|x\|}$,

$\nabla_\perp := (I_d - x_n x_n^\intercal)\nabla$ for $x \neq 0$ and $\nabla_\perp := \nabla$ for $x = 0$,

the orthogonal projection of ∇ on the tangent space $x^\perp$.



Moreover, if assumption 2) hold, then **any stationary density of p_∞ is rotation-invariant** on $\mathbb{R}^d$.

Gruhlke et al. (2026) in *ICLR*

Dynamics on d-spheres

Proposition (Gruhlke, Resseguier, Talla. 2026)

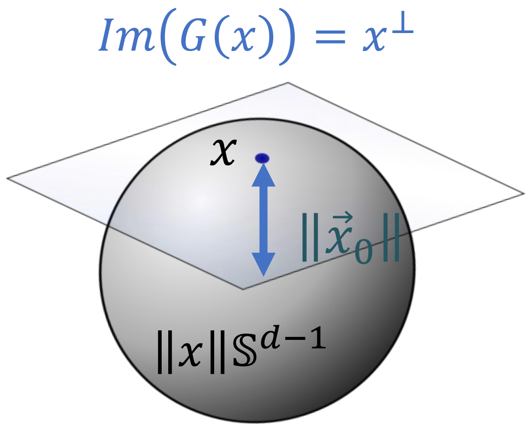
Distribution of the norms: Under 1), for all $t \geq 0$, $\|\vec{x}_s\| = \|\vec{x}_0\|$

Distribution of the directions (Fokker Planck equation)**:**

Under 1) and 2), let $p_0^n \in \mathcal{C}^2(\mathbb{S}^{d-1})$,

$$\frac{\partial p_s^n}{\partial s}(x^n) = \nabla_\perp \cdot \left(\frac{1}{2} \Sigma(x^n) \nabla_\perp p_s^n(x^n) \right), x^n \in \mathbb{S}^{d-1},$$

with $\Sigma(x^n) := G(x^n) G(x^n)^\intercal$, $x^n = \frac{x}{\|x\|}$,



- The Fokker Planck equation on $\mathbb{S}^{d-1}$ has **unique density solution** $\boldsymbol{p_t^n} \in \boldsymbol{\mathcal{C}^2}(\mathbb{S}^{\boldsymbol{d-1}})$ for all $t > 0$.
- There is unique invariant measure p_∞^n of the Fokker Planck , the uniform distribution on the $\mathbb{S}^{\boldsymbol{d-1}}$, with density

$$p_\infty^n(x^n) = \frac{1}{|\mathbb{S}^{\boldsymbol{d-1}}|}, \left|\mathbb{S}^{\boldsymbol{d-1}}\right| \text{ the volume of } \mathbb{S}^{\boldsymbol{d-1}}.$$

Gruhlke et al. (2026) in *ICLR*

Distribution in the latent space

Theorem 2 (Gruhlke, Resseguier, Talla. 2026)

Let $D = \mathbb{R}^d_*, \mathrm{d} > 1, \vec{x}_0 \sim p_0 \in \mathcal{C}^2(D), p_{|\cdot|}$ the density of $\| \vec{x}_0 \|$,

Under 1) and 2),

➢ The Fokker-Planck equation define on D has the **stationary distribution**

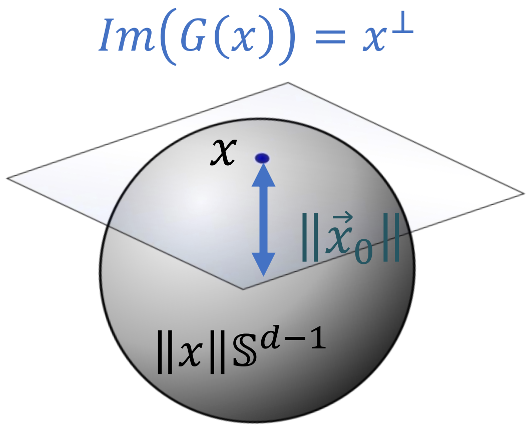
$$p_\infty(x) = p_{|\cdot|}(\| x \|) \frac{1}{\| x \|^{d-1} |\mathbb{S}^{\boldsymbol{d-1}}|}.$$

➢ There exist $\alpha = \alpha(G, d) > 0$ such that

$$\| p_t - p_\infty \|^2_{L^2(\mathbb{R}^d)} \leq \exp(-\alpha t) \| p_0 - p_\infty \|^2_{L^2(\mathbb{R}^d)},$$

where the convergence rate α is given by :

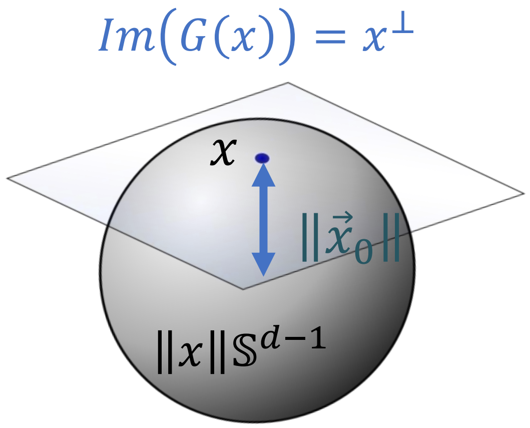
$$\alpha(G, d) = (d-1) \min_{(x,y) \in S} \| G^T(x) y \|^2, S = \{(x,y) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} | x \perp y\}.$$



Gruhlke et al. (2026) in *ICLR*

Distribution in the latent space

Corollary (Gruhlke, Resseguier, Talla. 2026)



Under 1) and 2),

- $\vec{x}_T^n = \frac{\vec{x}_T}{\|\vec{x}_T\|} \xrightarrow{\mathcal{L}} U^n \sim \mathcal{U}(\mathbb{S}^{d-1})$ when $T \to +\infty$
- $\|\vec{x}_T\| = \|\vec{x}_0\|$
- $\| \vec{x}_T \|$ and $\vec{x}_T^n$ are **asymptotically independent** when $\mathrm{T} \to +\infty$

we sample $\overleftarrow{x}_T$ as:

$$\overleftarrow{x}_T = \tilde{f}\left(\|\overleftarrow{x}_T^{\mathcal{N}}\|\right) \frac{\overleftarrow{x}_T^{\mathcal{N}}}{\|\overleftarrow{x}_T^{\mathcal{N}}\|}, \overleftarrow{x}_T^{\mathcal{N}} \sim \mathcal{N}(0, Id),$$

$\tilde{f}\left(\|\overleftarrow{x}_T^{\mathcal{N}}\|\right)$ parameterized by the distribution the one-dimensional variable $\|\vec{x}_0\|$,

Gruhlke et al. (2026) in *ICLR*

Backward diffusion with a neural network

Proposition (Gruhlke, Resseguier, Talla. 2026)

Under 1), the reverse-time SDE is given by :

$$d\overleftarrow{x}_t = \mathrm{G}(\overleftarrow{x}_t)\mathrm{G}(\overleftarrow{x}_t)^\intercal \nabla_{\overleftarrow{x}_t} \log p_{T-t}(\overleftarrow{x}_t)dt + \mathrm{G}(\overleftarrow{x}_t) \circ d\overleftarrow{B}_t$$

Gruhlke et al. (2026) in *ICLR*

Backward diffusion with a neural network

Proposition (Gruhlke, Resseguier, Talla. 2026)

Under 1), the reverse-time SDE is given by :

$$d\overleftarrow{x}_t = \mathrm{G}(\overleftarrow{x}_t)\,\underbrace{\mathrm{G}(\overleftarrow{x}_t)^{\intercal}\nabla_{\overleftarrow{x}_t}\log p_{T-t}(\overleftarrow{x}_t)}_{\approx a_\theta\left(\overrightarrow{x}_t, T-t\right)}\,dt + \mathrm{G}(\overleftarrow{x}_t)\circ d\overleftarrow{B}_t$$

Goal: find a_θ which maximize the likelihood of the observed data $\vec{x}_0$: $p_0(x|\theta)$

>> Too difficult… So, let's maximize the ELBO!

Gruhlke et al. (2026) in *ICLR*

Backward diffusion with a neural network

Proposition (Gruhlke, Resseguier, Talla. 2026)

Under 1), the reverse-time SDE is given by :

$$d\overleftarrow{x}_t = \mathrm{G}(\overleftarrow{x}_t)\, \underbrace{\mathrm{G}(\overleftarrow{x}_t)^\intercal \nabla_{\overleftarrow{x}_t} \log p_{T-t}(\overleftarrow{x}_t)}_{\approx a_\theta\left(\overrightarrow{x}_t, T-t\right)}\, dt + \mathrm{G}(\overleftarrow{x}_t) \circ d\overleftarrow{B}_t$$

Goal: find a_θ which maximize the likelihood of the observed data $\vec{x}_0$: $p_0(x|\theta)$

>> Too difficult… So, let's maximize the ELBO!

Theorem 3 (Gruhlke, Resseguier, Talla. 2026)

Under 1), we have this ELBO

$$p_0(x|\theta) \geq \mathcal{E}_\infty(x|\theta) \coloneqq \mathbb{E}[\log p_0(\vec{x}_T)|\vec{x}_0 = x] - \int_0^T \mathbb{E}_{\vec{x}_t}\left[\frac{1}{2}\|a_\theta(\vec{x}_s, s)\|^2 + \nabla_{\overleftarrow{x}_s} \cdot \left(\mathrm{G}(\vec{x}_s) a_\theta(\vec{x}_s, s)\right)\right] ds$$

Gruhlke et al. (2026) in *ICLR*

Backward diffusion with a neural network

Proposition (Gruhlke, Resseguier, Talla. 2026)

Under 1), the reverse-time SDE is given by :

$$d\overleftarrow{x}_t = \mathrm{G}(\overleftarrow{x}_t)\,\underbrace{\mathrm{G}(\overleftarrow{x}_t)^\top \nabla_{\overleftarrow{x}_t} \log p_{T-t}(\overleftarrow{x}_t)}_{\approx a_\theta\left(\overrightarrow{x}_t, T-t\right)}\, dt + \mathrm{G}(\overleftarrow{x}_t) \circ d\overleftarrow{B}_t$$

Goal: find a_θ which maximize the likelihood of the observed data $\vec{x}_0$: $p_0(x|\theta)$

>> Too difficult… So, let's maximize the ELBO!

Theorem 3 (Gruhlke, Resseguier, Talla. 2026)

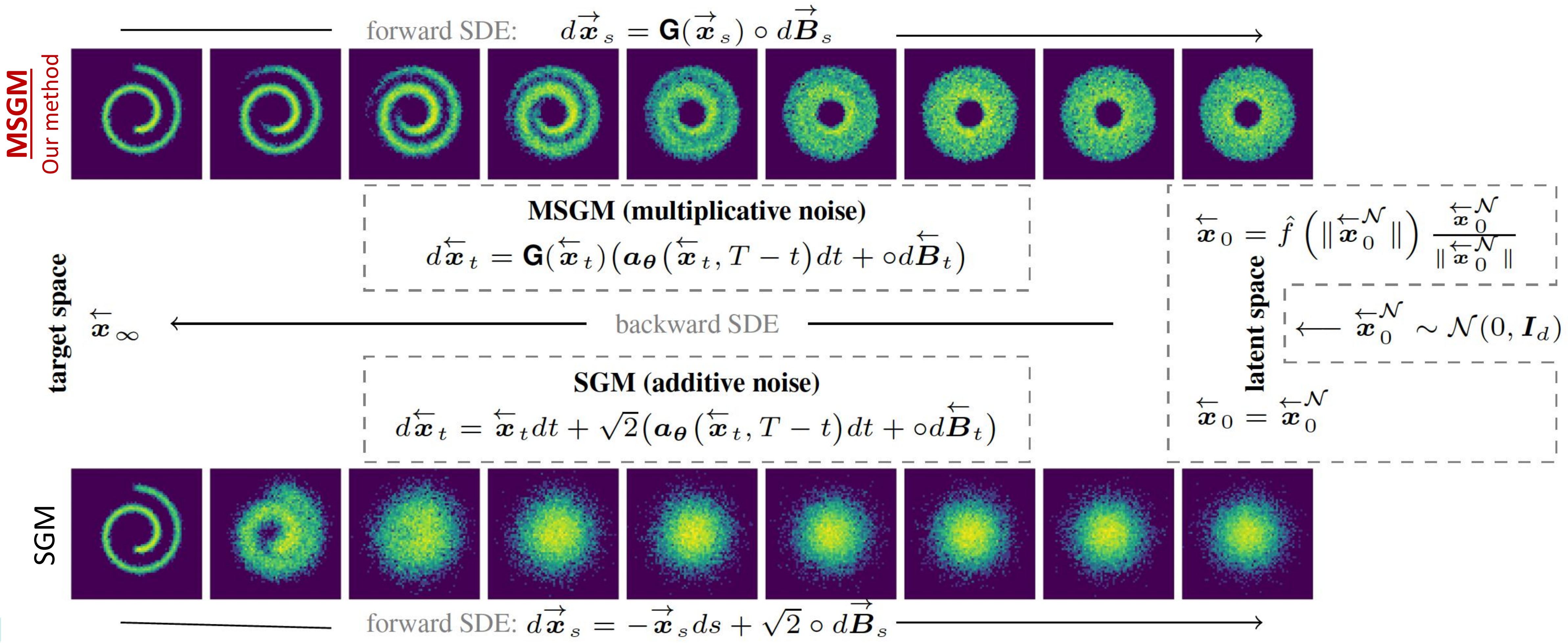
Under 1), we have this ELBO

$$p_0(x|\theta) \geq \mathcal{E}_\infty(x|\theta) \coloneqq \mathbb{E}[\log p_0(\vec{x}_T)|\vec{x}_0 = x] - \int_0^T \mathbb{E}_{\vec{x}_t}\left[\frac{1}{2}\|a_\theta(\vec{x}_s, s)\|^2 + \nabla_{\overleftarrow{x}_s} \cdot \left(\mathrm{G}(\vec{x}_s) a_\theta(\vec{x}_s, s)\right)\right] ds$$

In practice, $\nabla \cdot g = \mathbb{E}_v\left[v^\top \nabla_{\overleftarrow{x}_t} g^\top v\right]$ (sliced score matching)

Gruhlke et al. (2026) in *ICLR*

Method summary



Gruhlke et al. (2026) in *ICLR*

Rare events generation, in low dimension

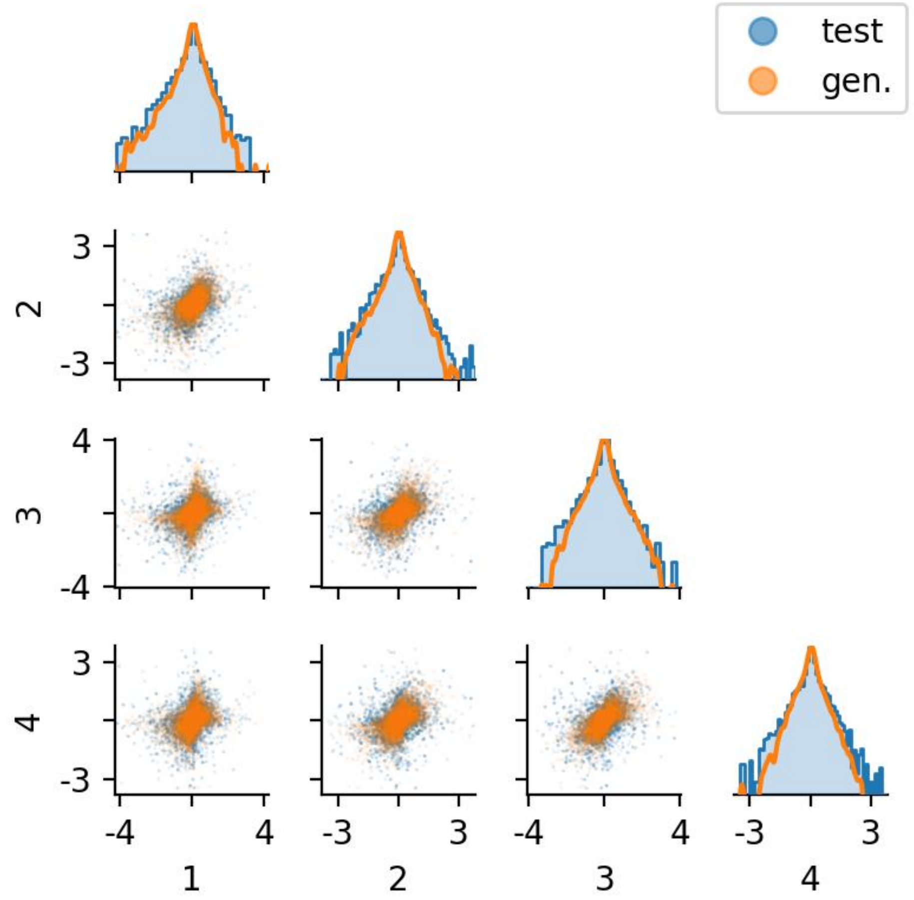
Dense tensor G^k $\qquad$ $M^k_{ij} \sim \mathcal{N}(0,1)$ (iid) and $G^k = \frac{1}{2}\left(M^k - \left(M^k\right)^{\top}\right)$

Measure vorticity fields

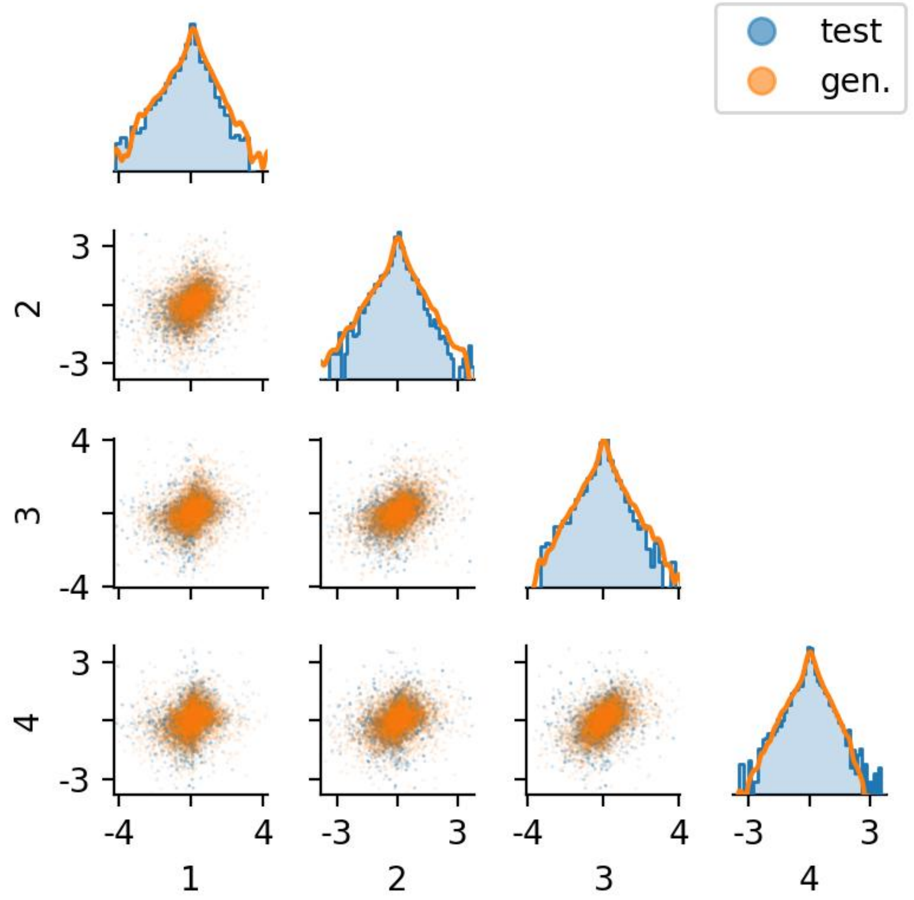
16-dimensional data points from PIV

1024 data train points only

SGM
(state-of-the-art)



MSGM
(our method)



Gruhlke et al. (2026) in *ICLR*

Rare events generation, in low dimension

Dense tensor G^k

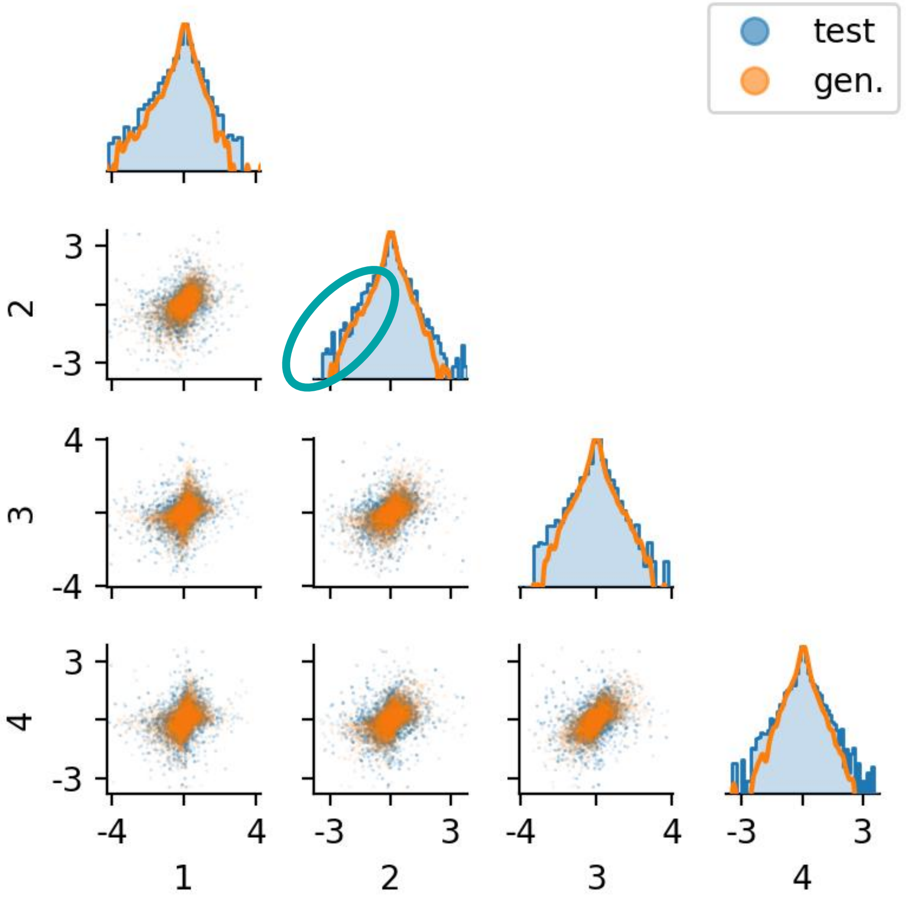
$M_{ij}^k \sim \mathcal{N}(0,1)$ (iid) and $G^k = \frac{1}{2}\left(M^k - (M^k)^\top\right)$

Measure vorticity fields

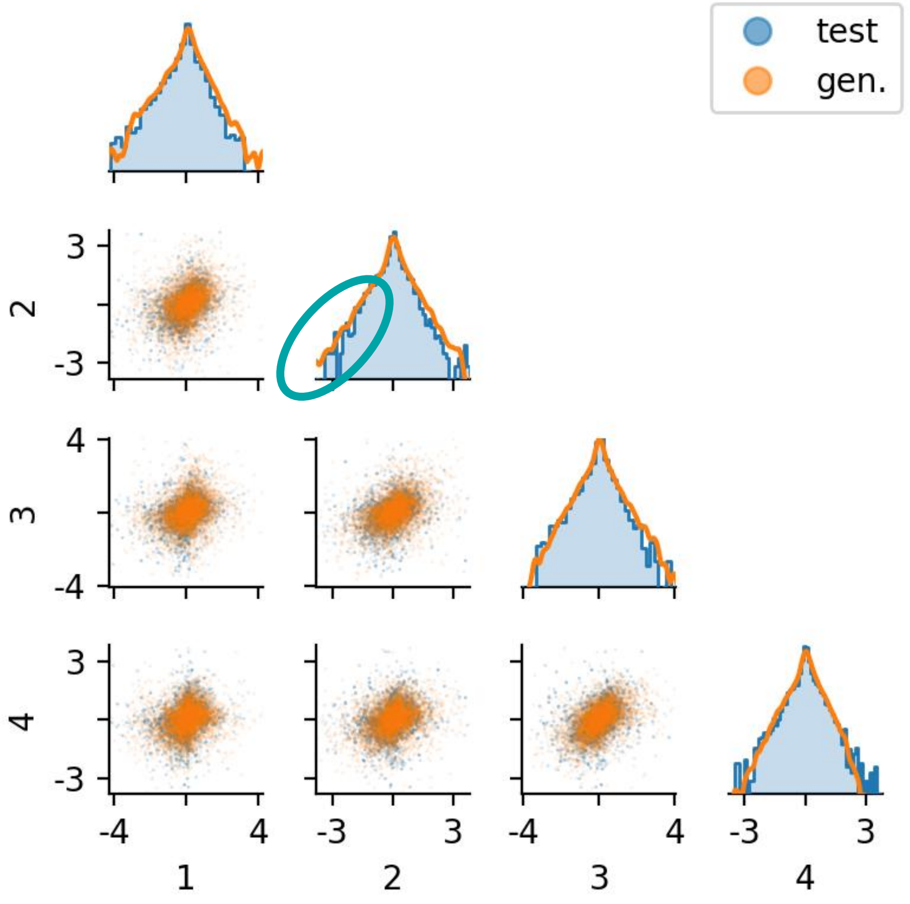
16-dimensional data points from PIV

1024 data train points only

SGM
(state-of-the-art)



MSGM
(our method)



Gruhlke et al. (2026) in *ICLR*

Rare events generation, in low dimension

Dense tensor G^k

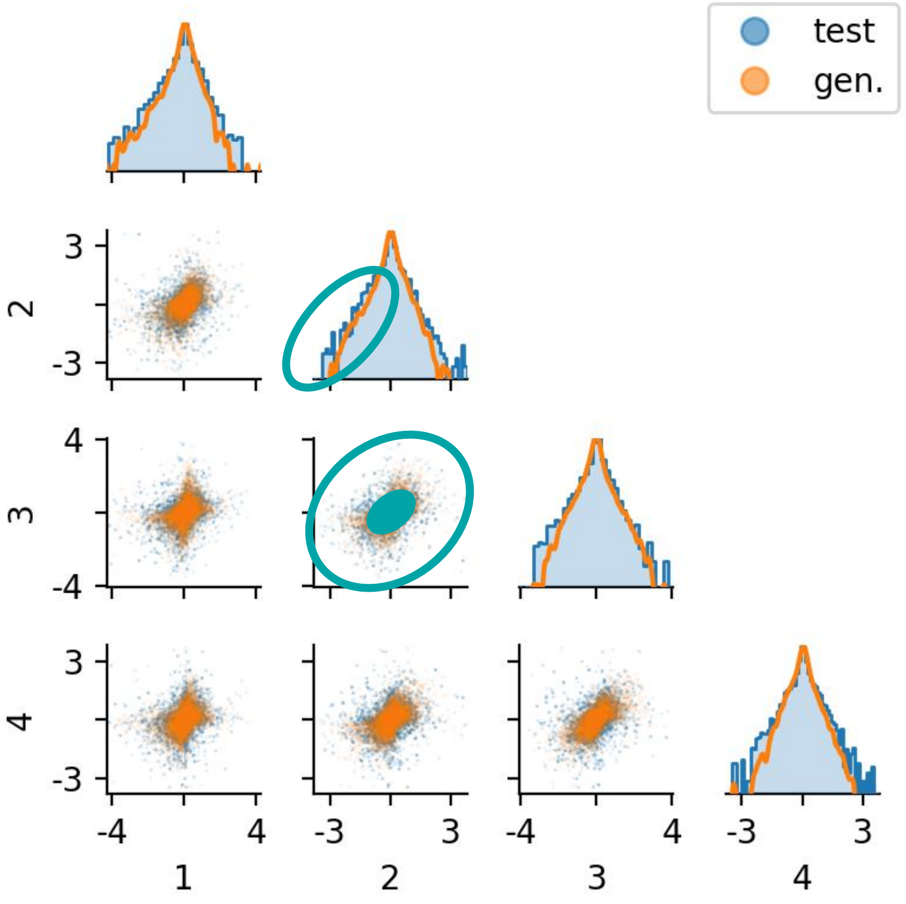
$M^k_{ij} \sim \mathcal{N}(0,1)$ (iid) and $G^k = \frac{1}{2}\left(M^k - (M^k)^\top\right)$

Measure vorticity fields

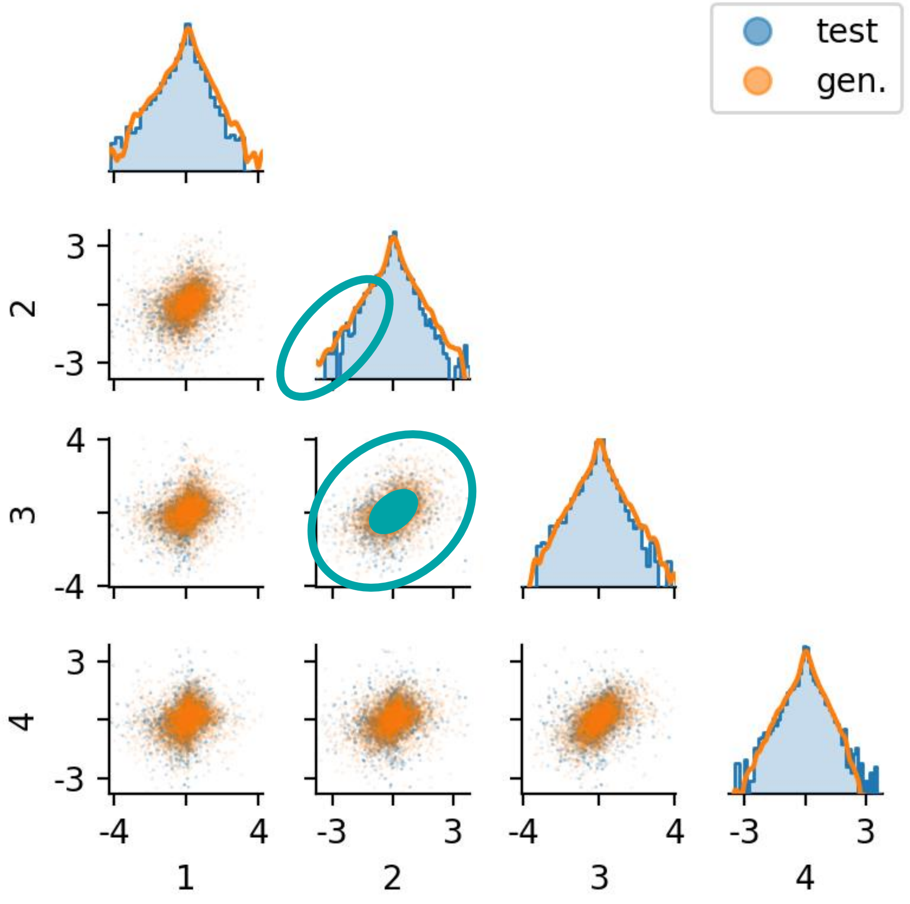
16-dimensional data points from PIV

1024 data train points only

SGM
(state-of-the-art)



MSGM
(our method)



Gruhlke et al. (2026) in *ICLR*

Rare events generation, in low dimension

Dense tensor G^k

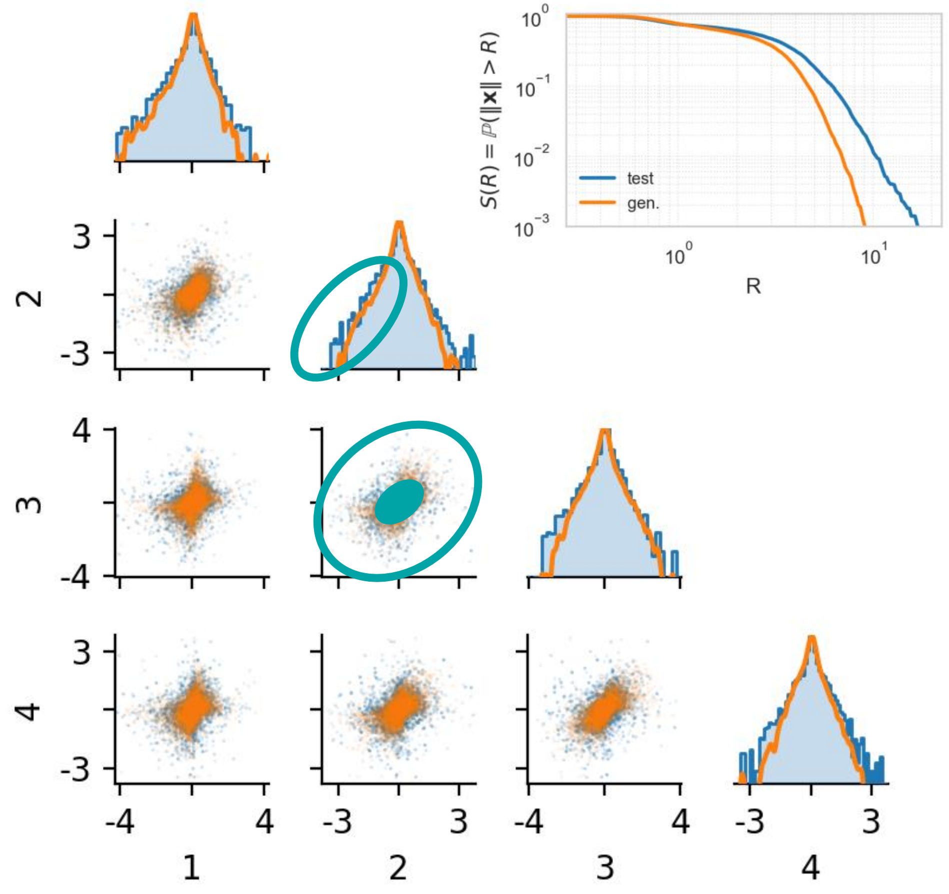
$M_{ij}^k \sim \mathcal{N}(0,1)$ (iid) and $G^k = \frac{1}{2}\left(M^k - (M^k)^\top\right)$

Measure vorticity fields

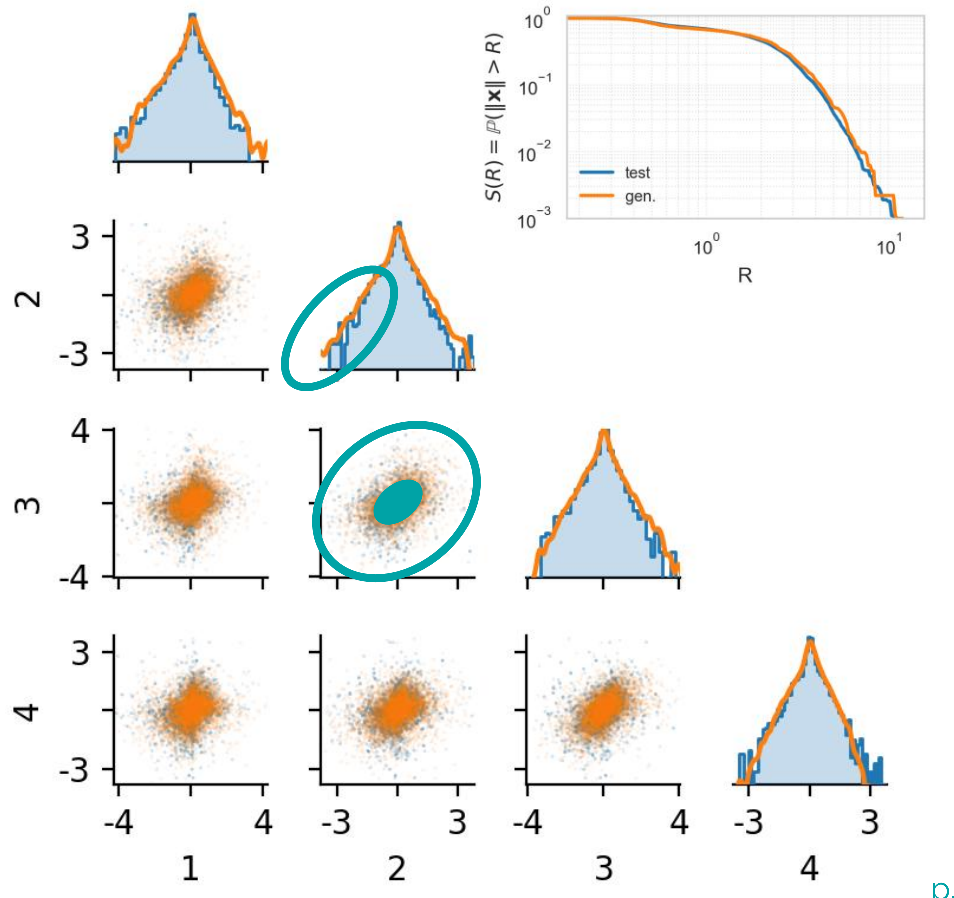
16-dimensional data points from PIV

1024 data train points only

SGM (state-of-the-art)



MSGM (our method)



Gruhlke et al. (2026) in *ICLR*

Rare events generation, in low dimension

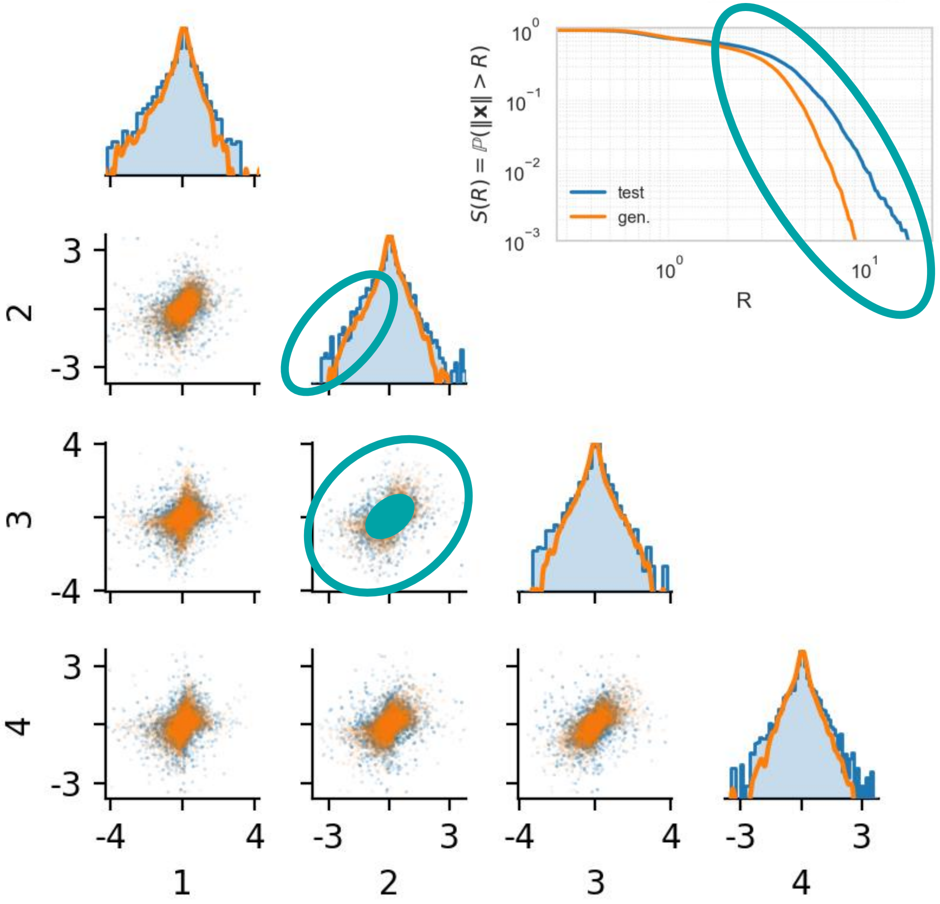
Dense tensor G^k $\qquad$ $M^k_{ij} \sim \mathcal{N}(0,1)$ (iid) and $G^k = \frac{1}{2}\left(M^k - \left(M^k\right)^\top\right)$

Measure vorticity fields

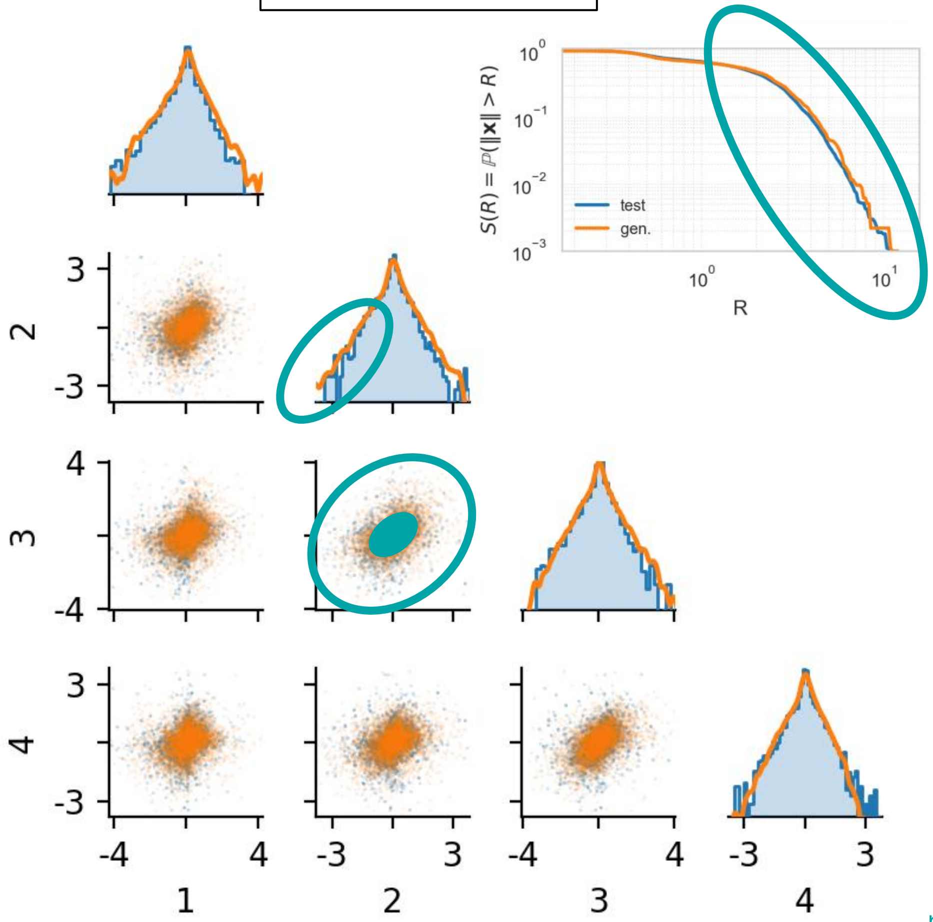
16-dimensional data points from PIV

1024 data train points only

SGM
(state-of-the-art)



MSGM
(our method)



Gruhlke et al. (2026) in *ICLR*

Rare events generation, in low dimension

Dense tensor G^k

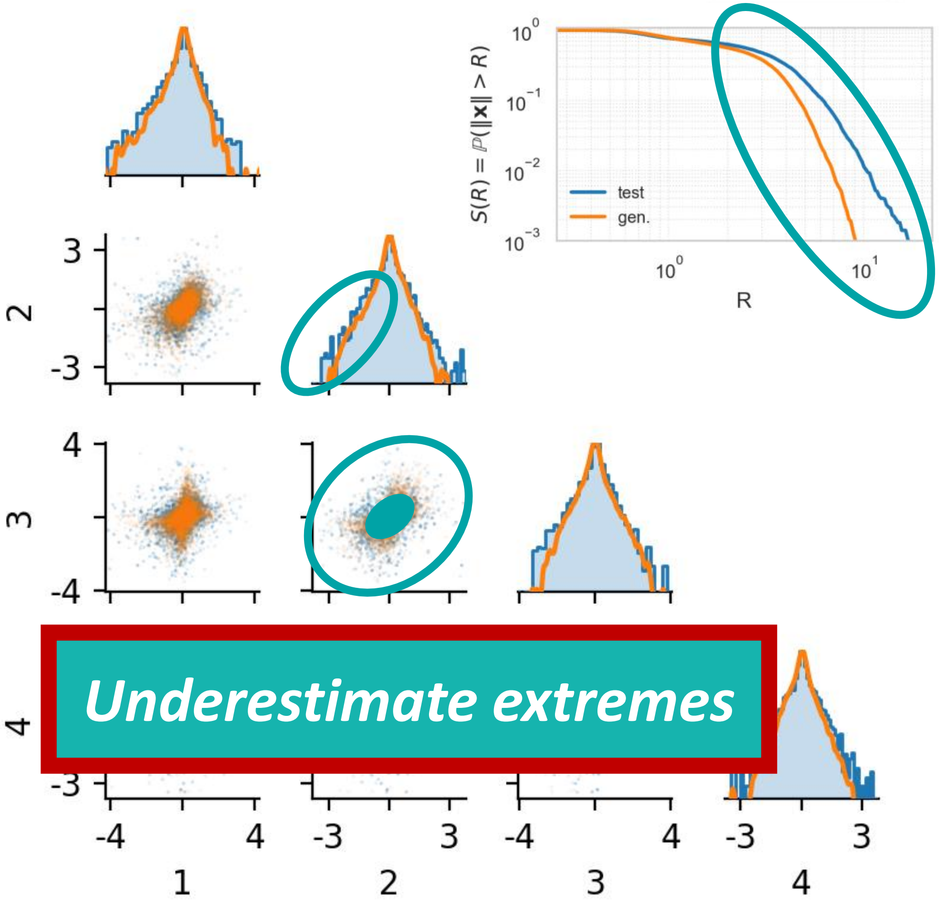
$M_{ij}^k \sim \mathcal{N}(0,1)$ (iid) and $G^k = \frac{1}{2}\left(M^k - \left(M^k\right)^{\top}\right)$

Measure vorticity fields

16-dimensional data points from PIV

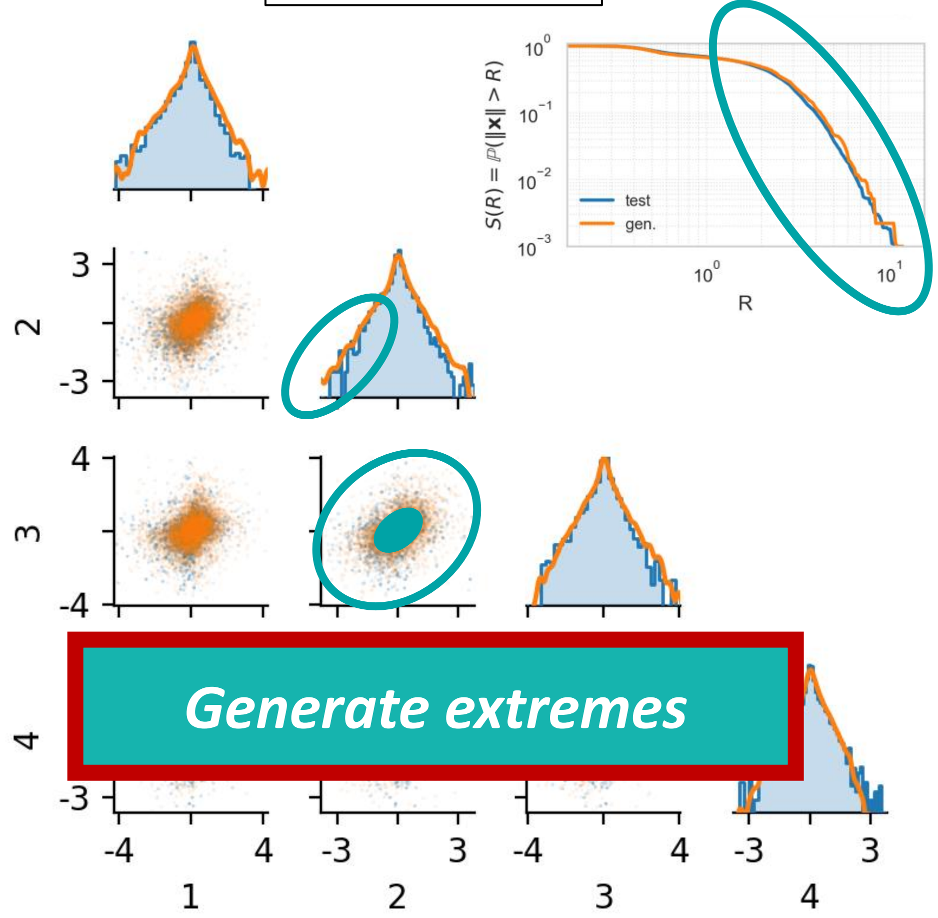
1024 data train points only

SGM
(state-of-the-art)



Underestimate extremes

MSGM
(our method)



Generate extremes

Gruhlke et al. (2026) in *ICLR*

Generation of fluid dynamics images (large dimension) with transport noise

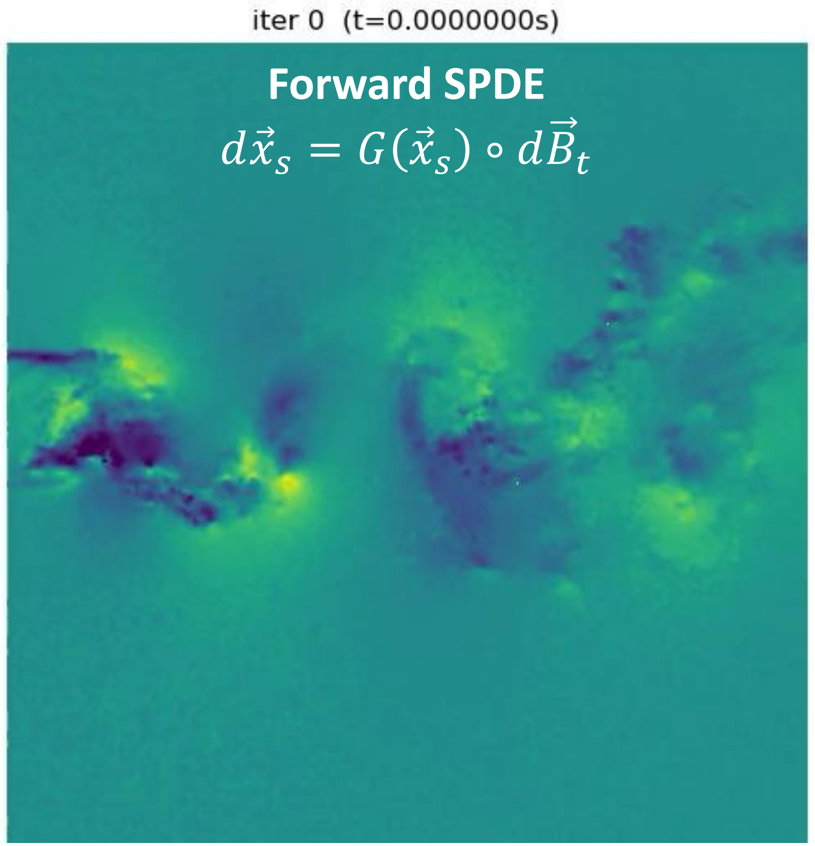
Sparse tensor G^k

Generation of fluid dynamics images (large dimension) with transport noise

Sparse tensor G^k

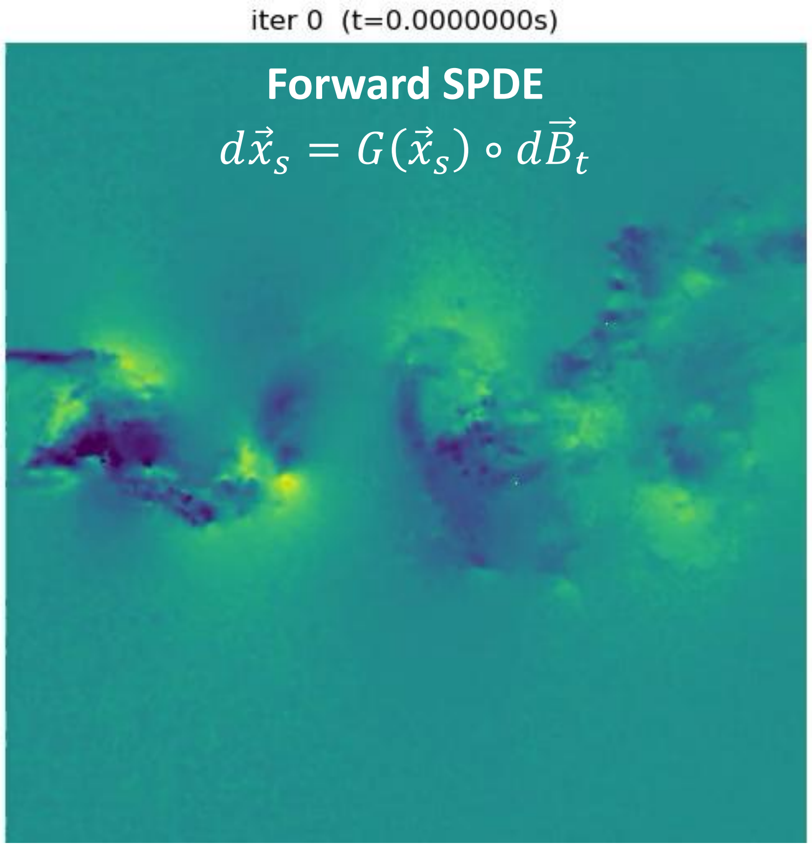
Generation of fluid dynamics images (large dimension) with transport noise

Sparse tensor G^k



Generation of fluid dynamics images (large dimension) with transport noise

Sparse tensor G^k



Open questions related about MSGM with sparse tensors

Assumption 2) does not holds!

2. Is a weaker assumption holds?
   ($\mathrm{Im}\big(G(x)\big) = x^{\perp}$, for almost every x in $\mathbb{R}^d$)
   Is it sufficient?
3. Convergence of the forward SPDE?
   To which distribution?
4. For convergence to weak white noise,
   statistical physics suggests additional constrains on the transporting Q-Wiener process
5. Speed of convergence? as $t^{-\beta}$?
6. Accelerate convergence with (skew-symmetric) drift,
   possibly a transport-based drift
   $d\vec{x}_s = F\vec{x}_s dt + G(\vec{x}_s) \circ d\vec{B}_t$

MSGM conclusion

- **New generative model paradigm**
- **General algorithm proposed**
  - Simple sampler for non-Gaussian latent vectors $\overleftarrow{x}_0$
  - (scaled) score learned by sliced score matching (SSM)
  - Allow both SDE and ODE denoising
- **Theoretical results proofed** (theorems):
  - Fokker-Planck equation of forward diffusion & its invariant measures
  - Diffusion convergence (exponentially fast) to a white noise in the weak sense
  - Our score matching = maximizing the ELBO
- **Applications**
  - rare /events extremes : **MSGM need less data**
  - Image generation, with sparse tensors

Other open questions

MSGM outlooks

8. Analytic **solution sampler for the forward SDE** at finite time s
   to accelerate the training (as Orstein-Uhlenbeck sampling in classical SGM)
9. Analytic expression for the (scaled) **score,** $G(x)^\intercal \boldsymbol{\nabla_x \log p_s(x)}$ at finite time s
   to enable denoising score matching (DSM) instead of inaccurate sliced score matching (SSM)
10. Asymptotic results for $d \to +\infty$?
    e.g. from the free multiplicative Brownian motion theory
11. Multiplicative Flow Matching, with stochastic interpolants on the d-sphere geodesics
12. Score in Bayesian inverse problem / data assimilation

$$\nabla_x \log p_s(x|y) = \underbrace{\nabla_x \log p_s(x)}_{\text{learned}} + \underbrace{\nabla_x \log p(y|x_s)}_{=?} \quad \text{where} \quad \nabla_x \log p(y|x_0) = \Sigma_y^{-1}(y - Hx)$$

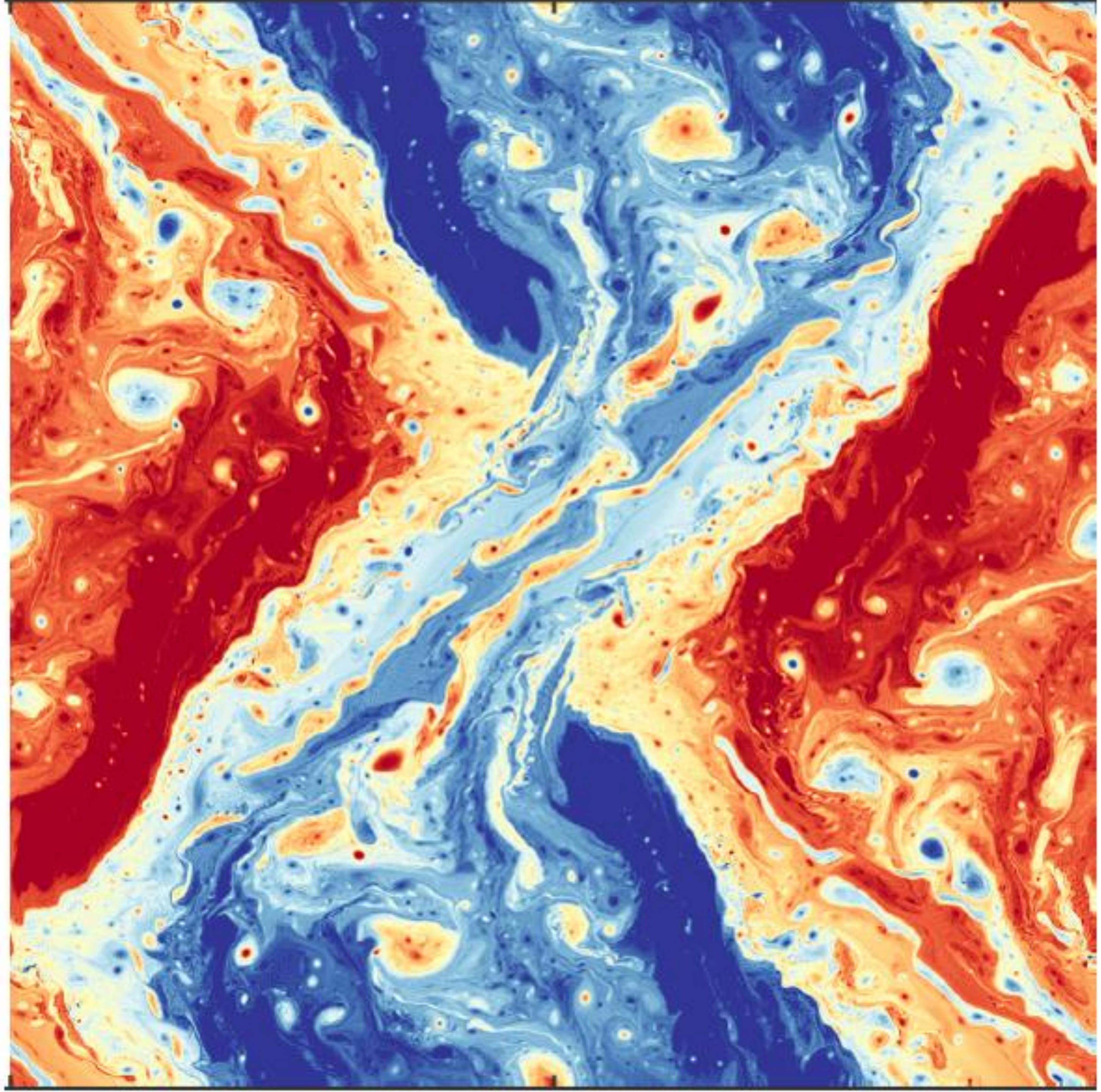
13. What about *nonlinear* Forward S(P)DE (e.g. Navier-Stokes LU) ?

$$d\vec{x}_s = F(\vec{x}_s)dt + G(\vec{x}_s) \circ d\vec{B}_t$$

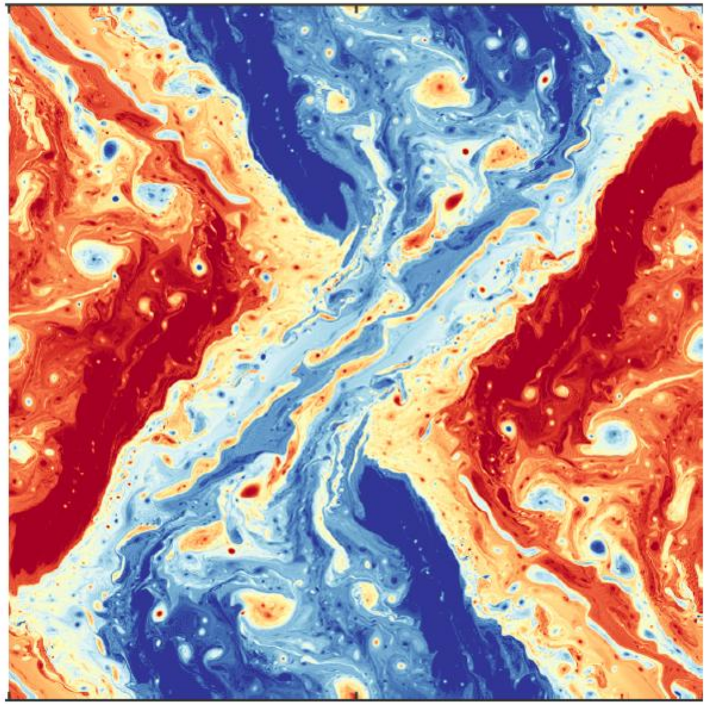
IV. Beyond classical transport noises

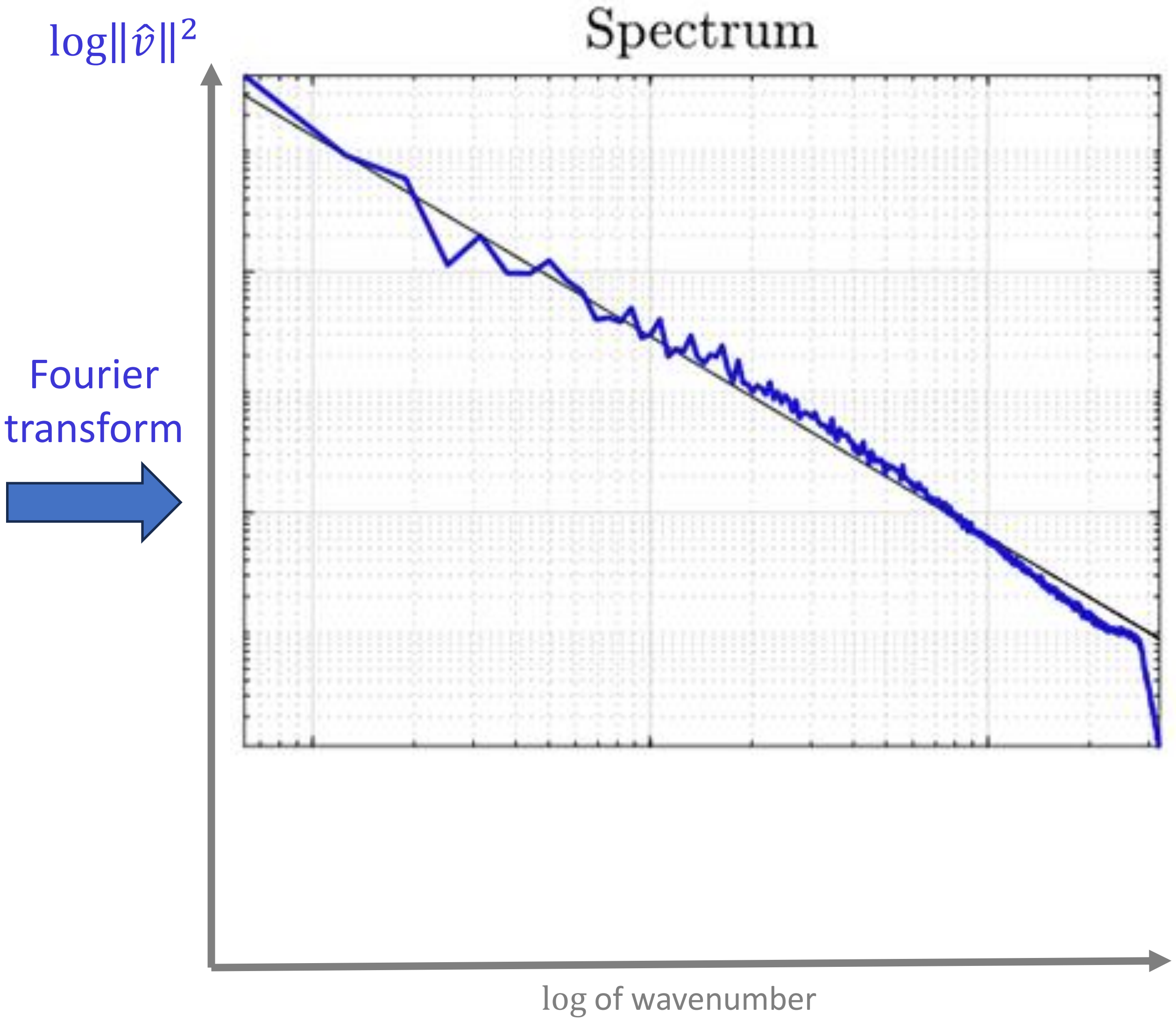
- Scale symmetry
- Gaussian processes multiscale in time
- Gaussian multiplicative chaos
- Open questions

In collaboration with : William Antolin, Carlos G Belinchon

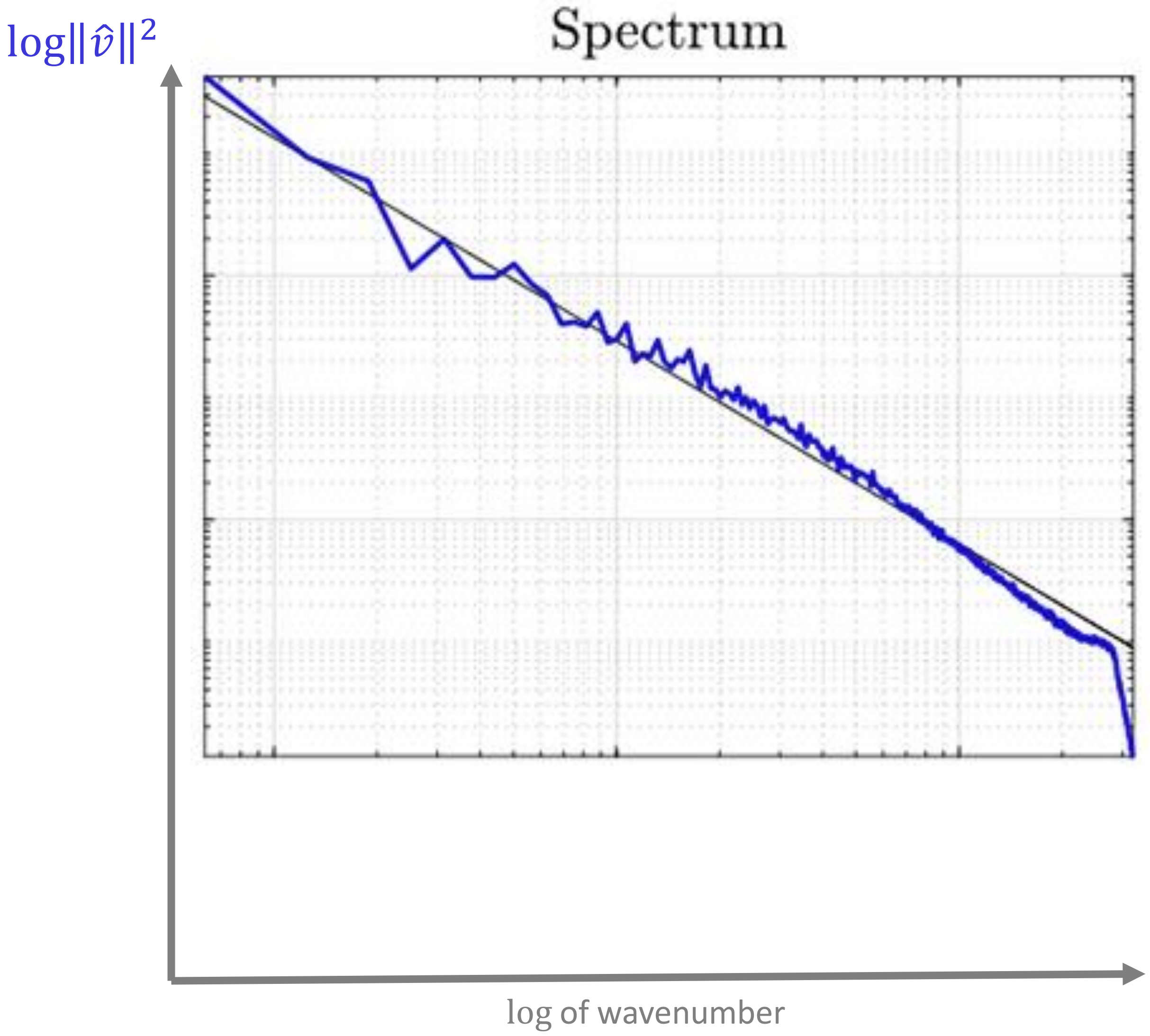


Scale symmetry

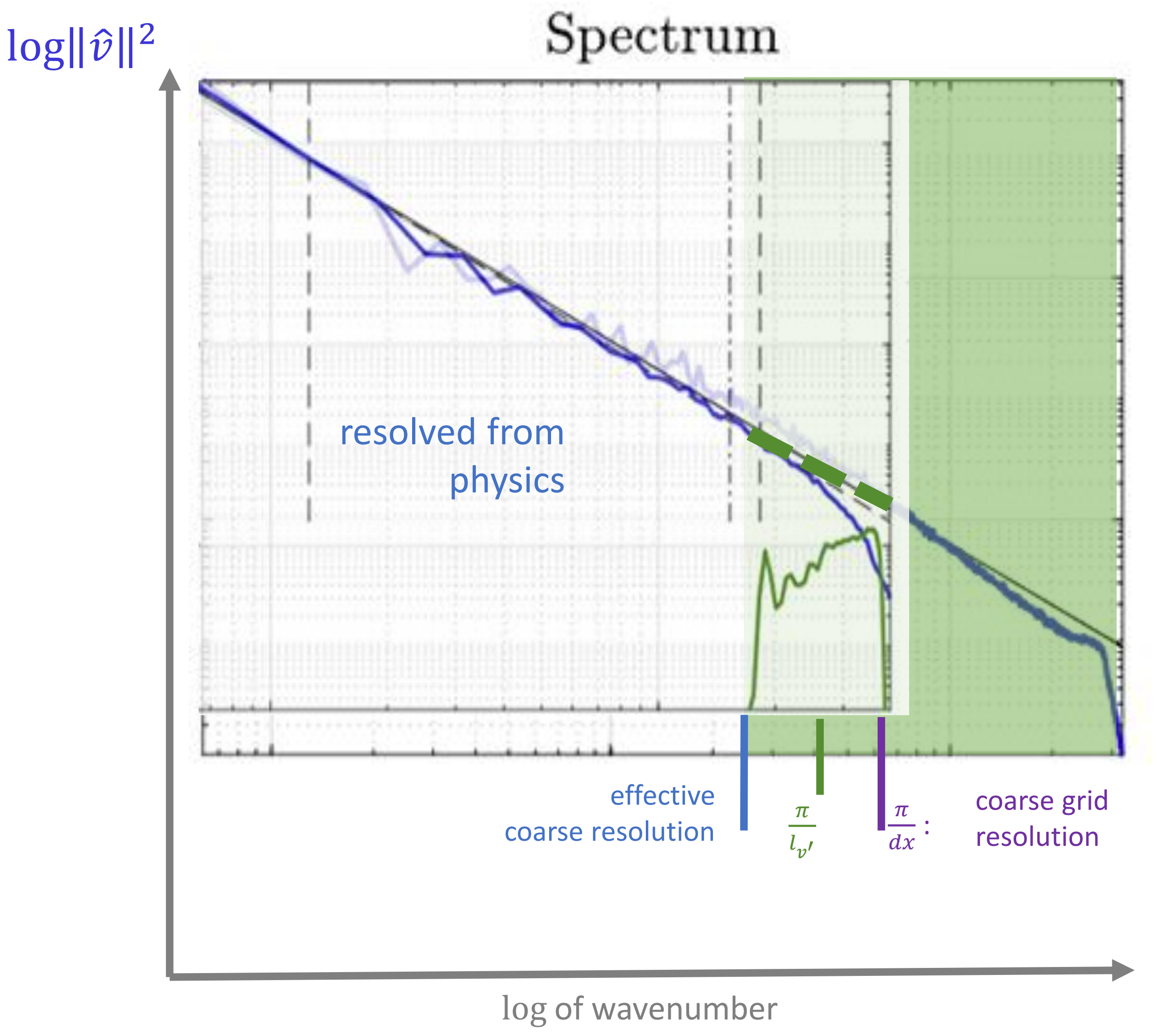




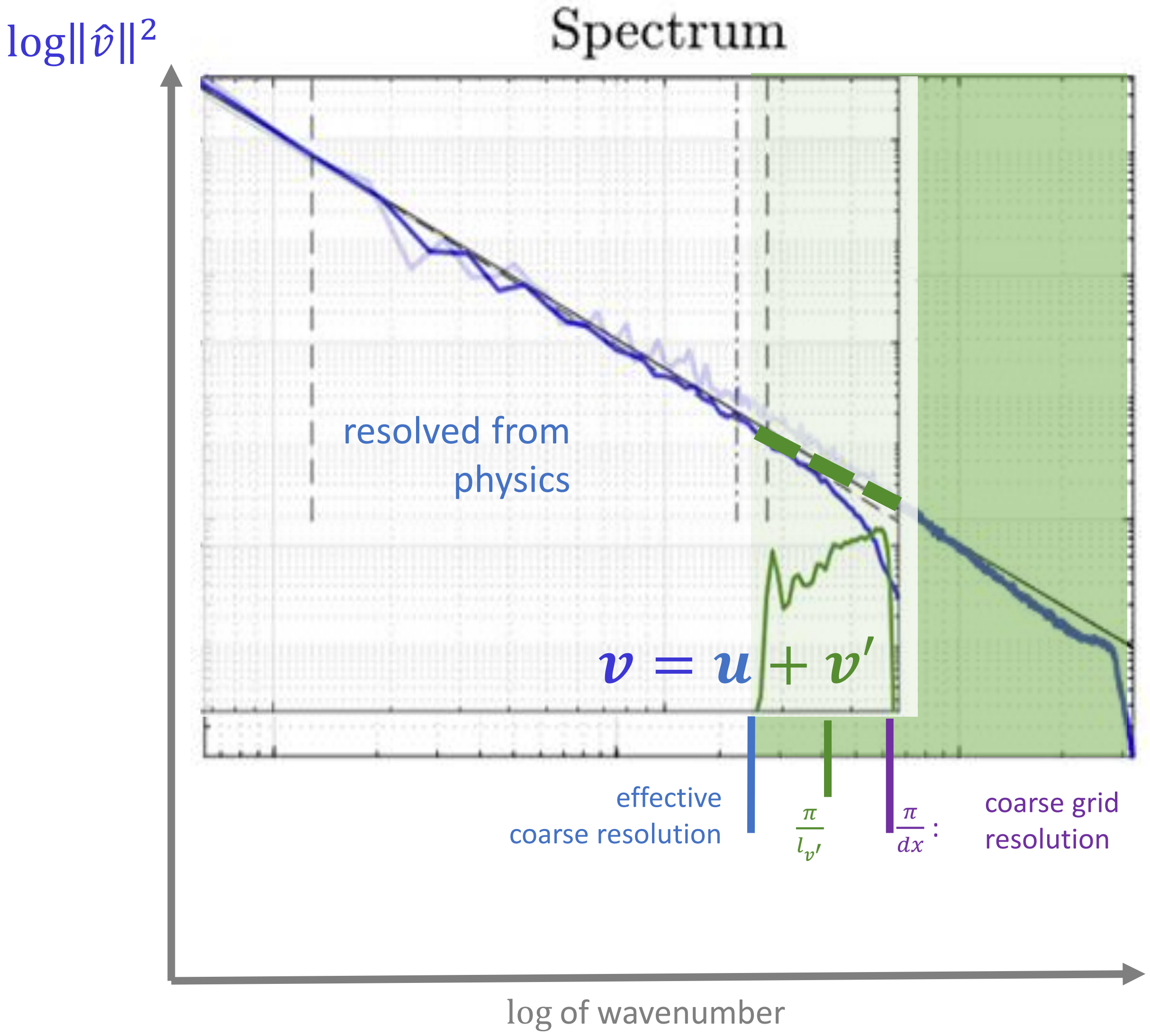
Scale symmetry



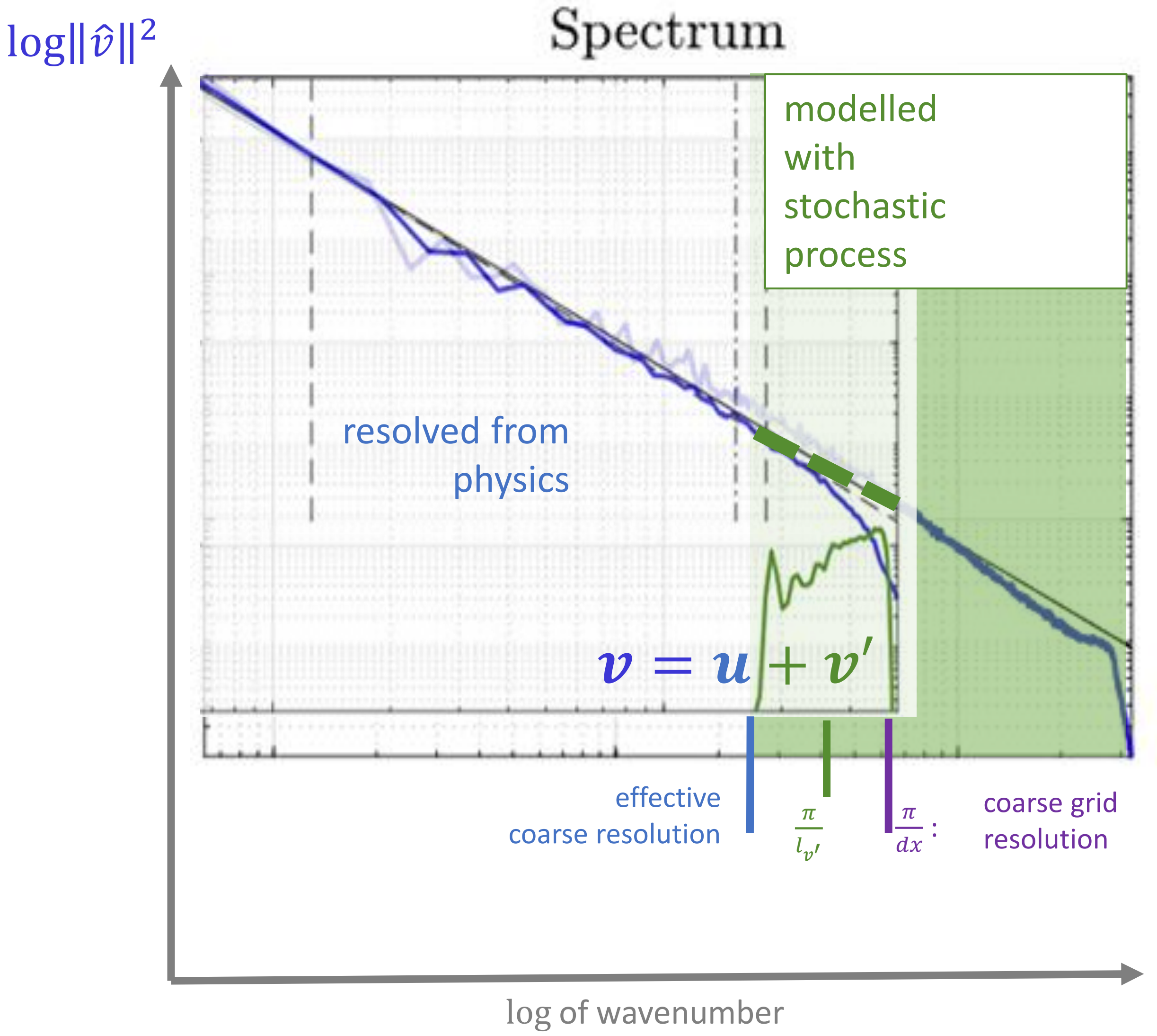
Scale symmetry



Scale symmetry



Scale symmetry



Scale symmetry

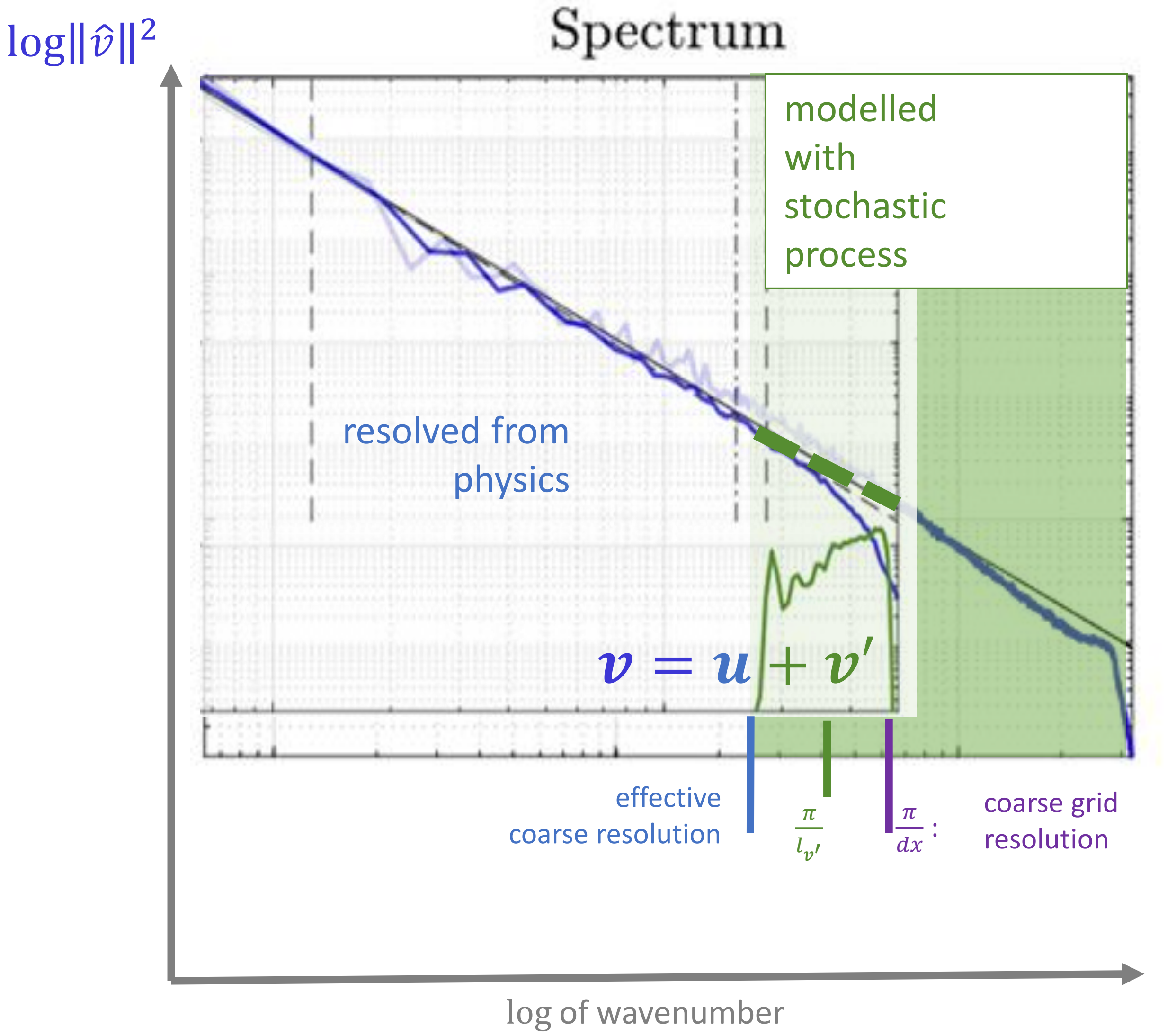
$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sigma(x) \circ \frac{dB_t}{dt}$$
$$= \nabla \times \left(\psi *\circ \frac{dB_t}{dt}\right)$$
$$= \sum_k ik \times \sqrt{\Gamma_1(k)}\, e^{ik\cdot x} \circ \frac{d\widehat{B_t}(k)}{dt}$$

increments of Q-Wiener process
(Gaussian, divergence-free, white wrt t)



Scale symmetry & two-way coupling

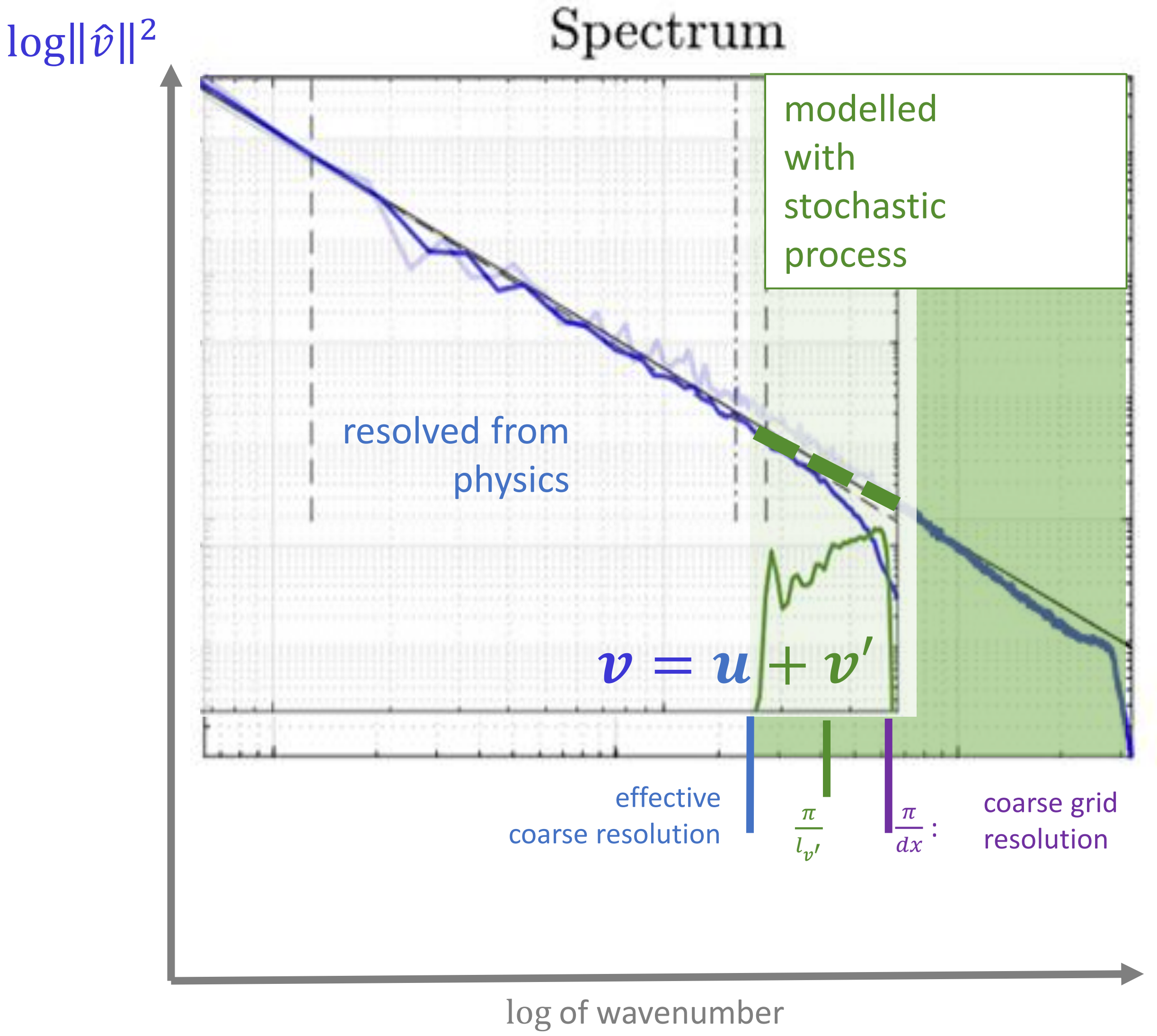
$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sigma(x) \circ \frac{dB_t}{dt}$$
$$= \nabla \times \left(\psi *\circ \frac{dB_t}{dt}\right)$$
$$= \sum_k ik \times \sqrt{\Gamma_1(k)}\, e^{ik\cdot x} \circ \frac{d\widehat{B_t}(k)}{dt}$$

increments of Q-Wiener process
(Gaussian, divergence-free, white wrt t)



Scale symmetry & two-way coupling

$$v = u + v'$$

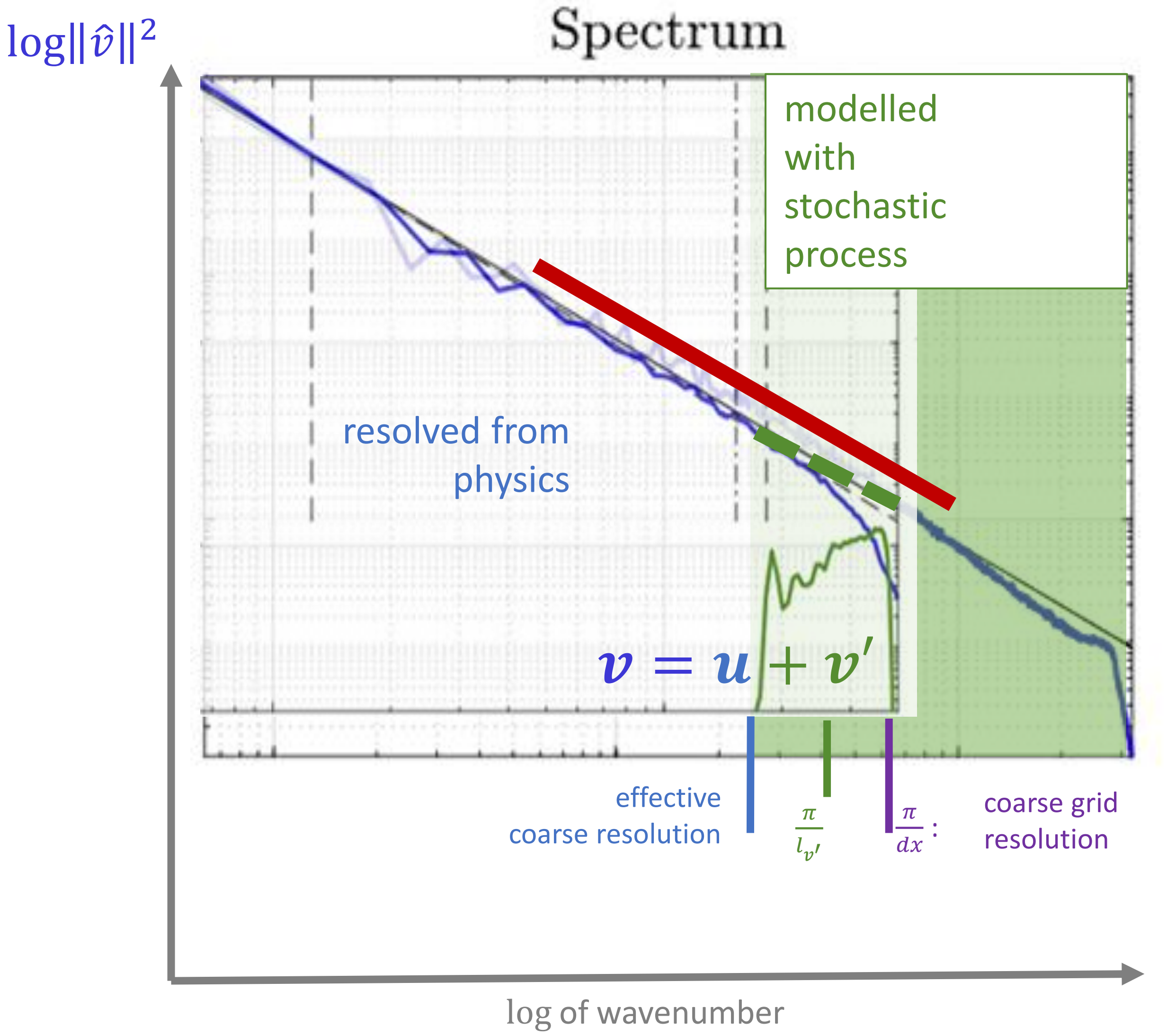
Resolved fluid velocity:

u

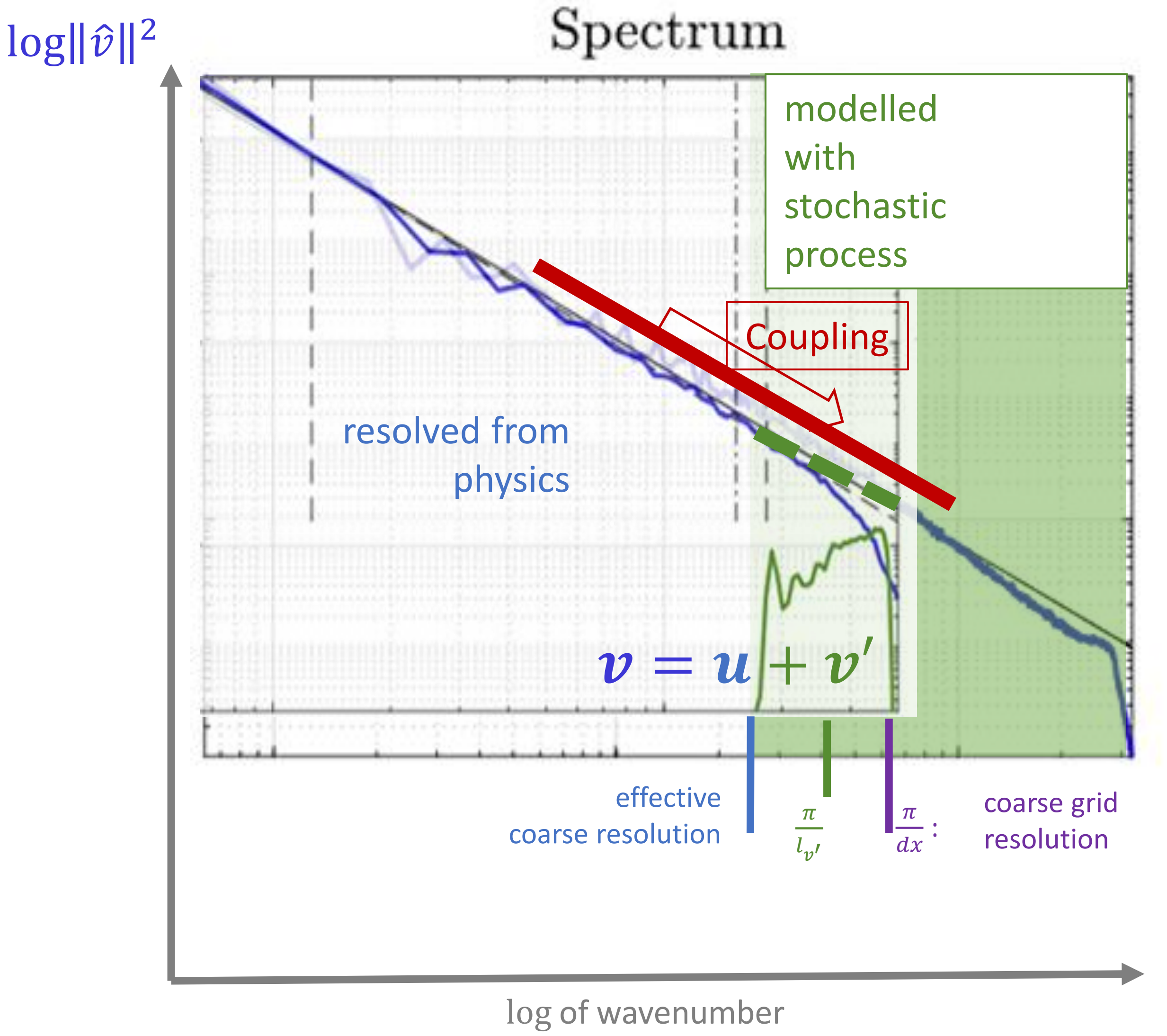
Unresolved fluid velocity:

$$v'(x,t) = \sigma(x) \circ \frac{dB_t}{dt}$$
$$= \nabla \times \left(\psi *\circ \frac{dB_t}{dt}\right)$$
$$= \sum_k ik \times \sqrt{\Gamma_1(k)}\, e^{ik\cdot x} \circ \frac{d\widehat{B_t}(k)}{dt}$$

increments of Q-Wiener process
(Gaussian, divergence-free, white wrt t)



Scale symmetry & two-way coupling



$$v = u + v'$$

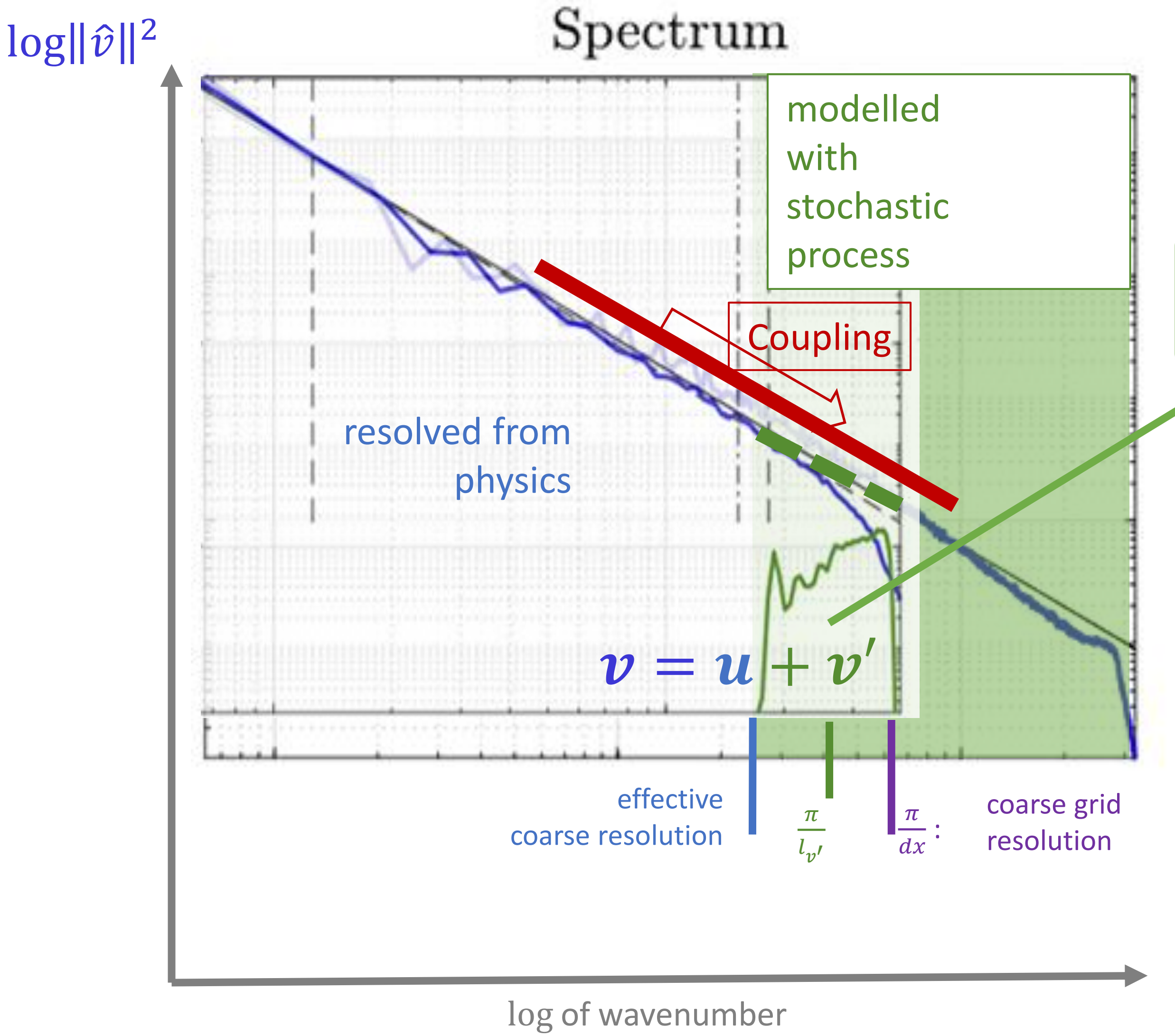
Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sigma(x) \circ \frac{dB_t}{dt}$$
$$= \nabla \times \left(\psi *\circ \frac{dB_t}{dt}\right)$$
$$= \sum_k ik \times \sqrt{\Gamma_1(k)}\, e^{ik\cdot x} \circ \frac{d\widehat{B_t}(k)}{dt}$$

increments of Q-Wiener process
(Gaussian, divergence-free, white wrt t)

Scale symmetry & two-way coupling



Residual spectrum Γ_1

$\Gamma_1^{\circ} = \text{cst.}\|\boldsymbol{k}\|^{-\rho} - \|\hat{\boldsymbol{u}}\|^2$

$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sigma(x) \circ \frac{dB_t}{dt}$$
$$= \nabla \times \left(\psi * \circ \frac{dB_t}{dt}\right)$$
$$= \sum_k ik \times \sqrt{\Gamma_1(k)}\, e^{ik\cdot x} \circ \frac{d\widehat{B_t}(k)}{dt}$$

increments of Q-Wiener process
(Gaussian, divergence-free, white wrt t)

Scale symmetry & two-way coupling

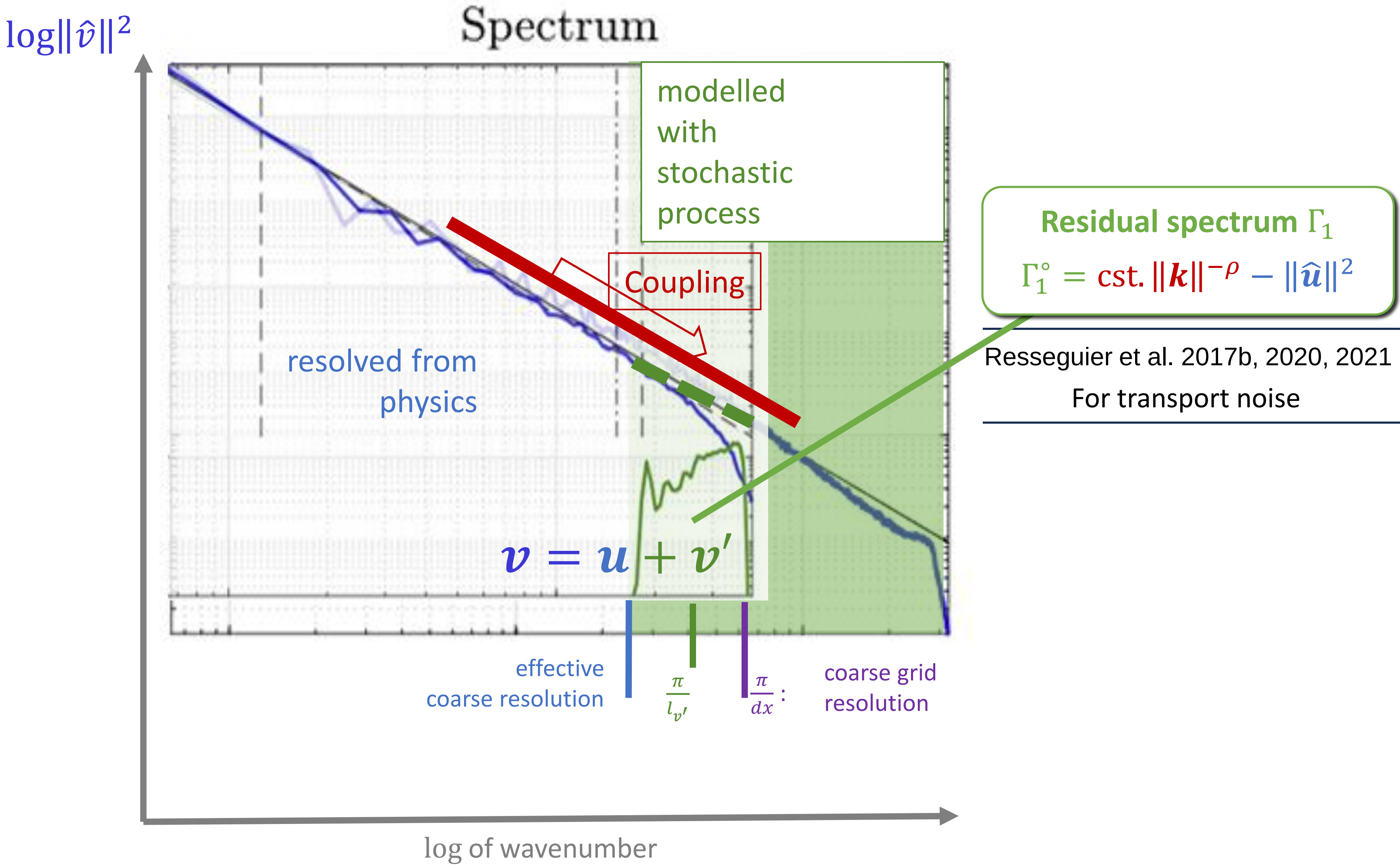
$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sigma(x) \circ \frac{dB_t}{dt}$$
$$= \nabla \times \left(\psi \ast\circ \frac{dB_t}{dt}\right)$$
$$= \sum_k ik \times \sqrt{\Gamma_1(k)}\, e^{ik\cdot x} \circ \frac{d\widehat{B_t}(k)}{dt}$$

increments of Q-Wiener process
(Gaussian, divergence-free, white wrt t)



Resseguier et al. 2017b, 2020, 2021
For transport noise

Scale symmetry & two-way coupling

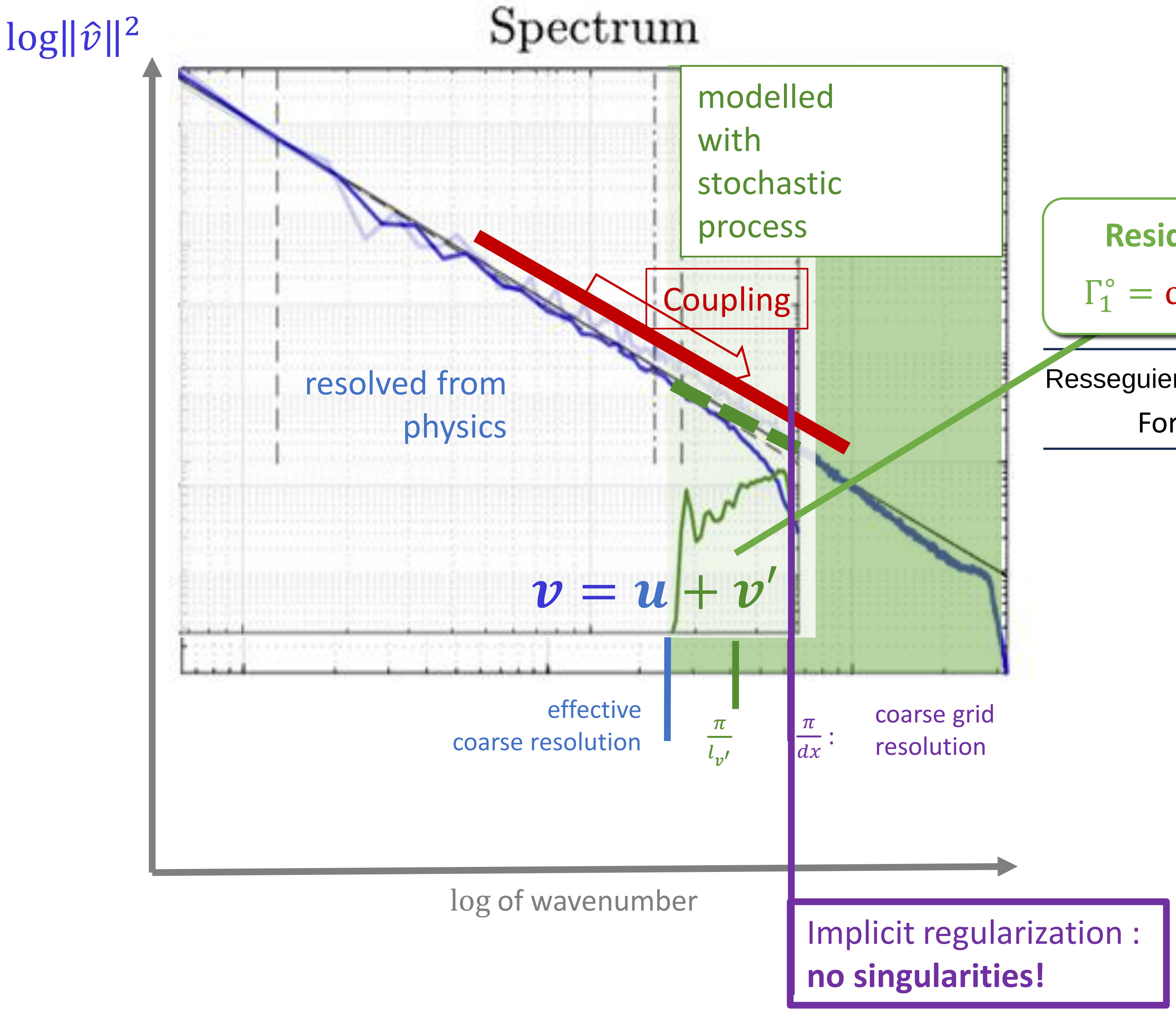
$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sigma(x) \circ \frac{dB_t}{dt}$$
$$= \nabla \times \left(\psi *\circ \frac{dB_t}{dt} \right)$$
$$= \sum_k ik \times \sqrt{\Gamma_1(k)}\, e^{ik\cdot x} \circ \frac{d\widehat{B_t}(k)}{dt}$$

increments of Q-Wiener process
(Gaussian, divergence-free, white wrt t)



Residual spectrum Γ_1

$$\Gamma_1^\circ = \text{cst.}\, \|\boldsymbol{k}\|^{-\rho} - \|\widehat{\boldsymbol{u}}\|^2$$

Resseguier et al. 2017b, 2020, 2021
For transport noise

Scale symmetry & two-way coupling

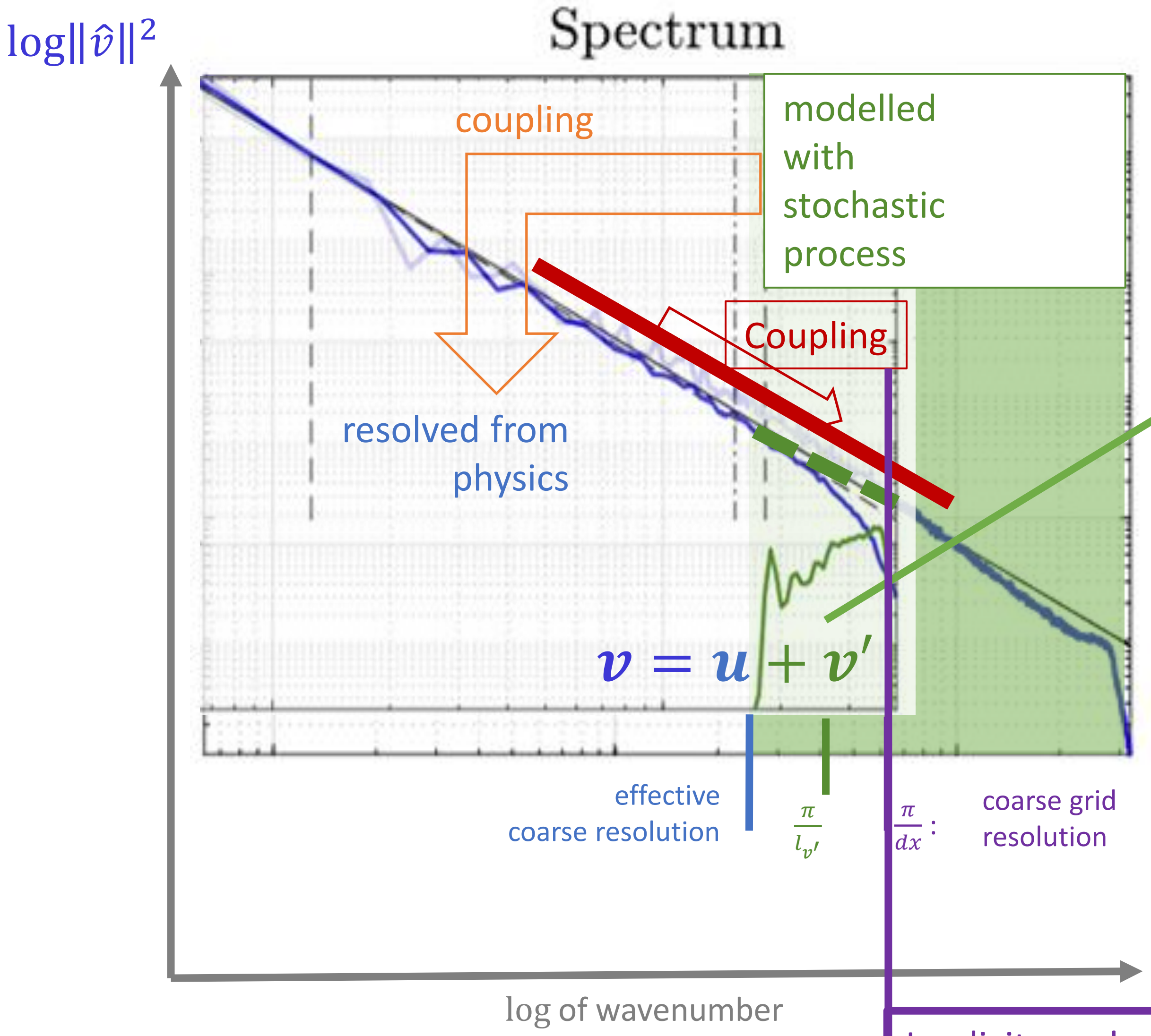
$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sigma(x) \circ \frac{dB_t}{dt}$$
$$= \nabla \times \left(\psi *\circ \frac{dB_t}{dt}\right)$$
$$= \sum_k ik \times \sqrt{\Gamma_1(k)}\, e^{ik\cdot x} \circ \frac{d\widehat{B_t}(k)}{dt}$$

increments of Q-Wiener process
(Gaussian, divergence-free, white wrt t)



Residual spectrum Γ_1

$$\Gamma_1^\circ = \text{cst.}\|\boldsymbol{k}\|^{-\rho} - \|\widehat{\boldsymbol{u}}\|^2$$

Resseguier et al. 2017b, 2020, 2021
For transport noise

Implicit regularization :
no singularities!

Scale symmetry & two-way coupling

Stochastic transport

v' injected in physics equation, through *transport noise* :

$$d_t u + (u dt + \sigma \circ dB_t) \cdot \nabla u = dF$$

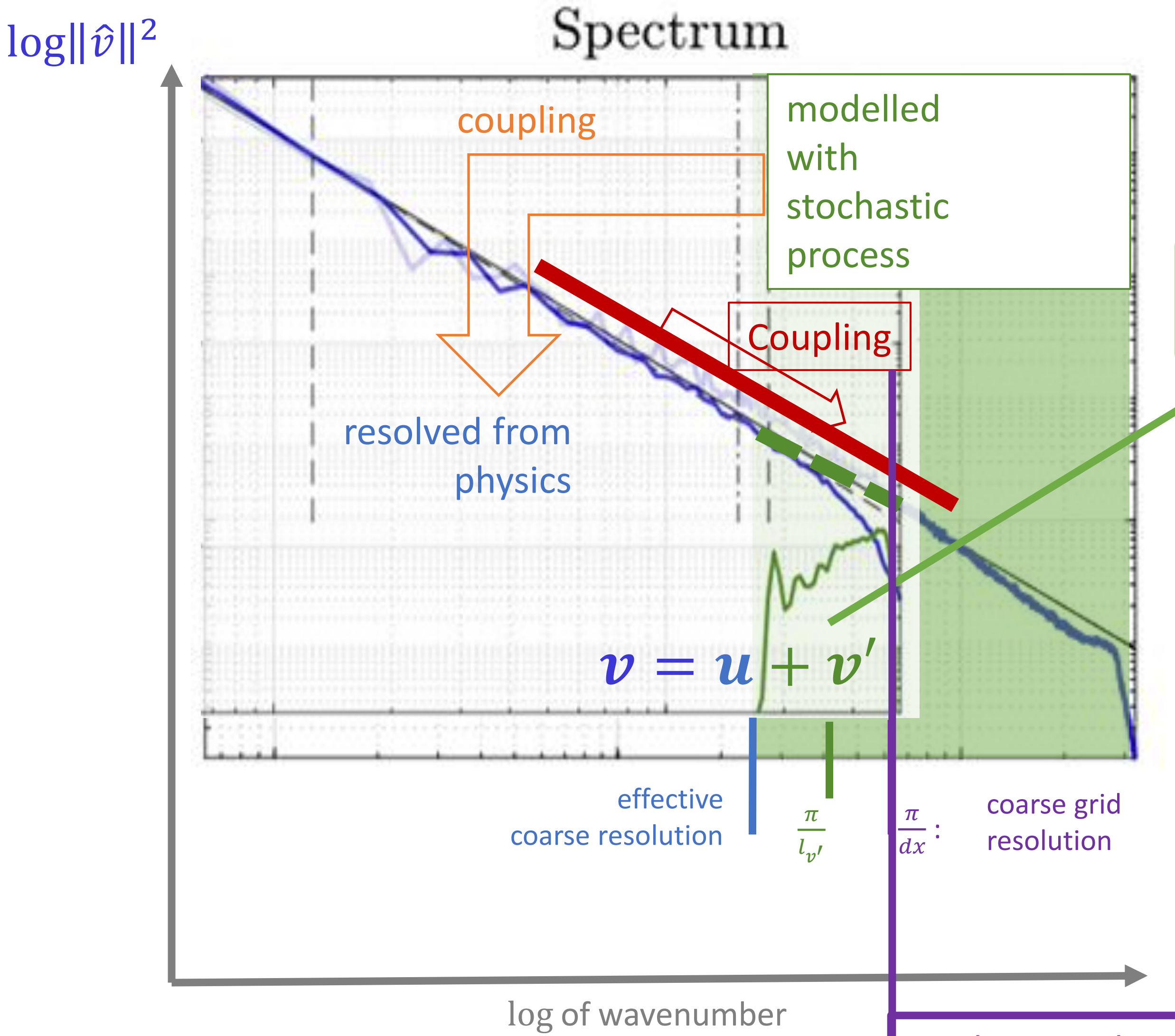
$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sigma(x) \circ \frac{dB_t}{dt} = \nabla \times \left(\psi *\circ \frac{dB_t}{dt}\right) = \sum_k ik \times \sqrt{\Gamma_1(k)}\, e^{ik\cdot x} \circ \frac{d\widehat{B_t}(k)}{dt}$$

increments of Q-Wiener process
(Gaussian, divergence-free, white wrt t)



Residual spectrum Γ_1

$$\Gamma_1^\circ = \text{cst.} \|\boldsymbol{k}\|^{-\rho} - \|\widehat{\boldsymbol{u}}\|^2$$

Resseguier et al. 2017b, 2020, 2021
For transport noise

Implicit regularization :
no singularities!

Simple noise $O_t(x)$: v' fractal in x
Usual transport noise

Gaussian noise

$$\text{Noise} = dN_t(x) = dB_t(x)$$

$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sigma(x) \circ \frac{dN_t}{dt} = \nabla \times \left(\psi *\circ \frac{dN_t}{dt} \right) = \sum_k ik \times \sqrt{\Gamma_1(k)}\, e^{ik\cdot x} \circ \frac{\widehat{dN_t}(k)}{dt}$$

increments of Q-Wiener process
(Gaussian, divergence-free, white wrt t)

Gaussian noise

$$\mathbb{E}\{\, dB_t(x)\, dB_t(x+\delta x)\} \propto \delta(\delta x)\, dt$$

Resseguier et al. 2017b, 2020, 2021
For transport noise

Full turbulence field

$$\frac{1}{dt}\mathbb{E}\left\|\widehat{dB_t}(k)\right\|^2 = 1$$

$$\mathbb{E}\left\|\widehat{v'}(k)\right\|^2 = \frac{1}{dt}\|\boldsymbol{k}\|\Gamma_1 \propto \left(1 + \frac{\|k\|^2}{k_0^2}\right)^{\frac{-\rho-d+1}{2}}$$

- $E(\|k\|) = \oint d\theta_k \|k\| \left\|\widehat{v'}(k)\right\|^2 \sim \text{cst.}\, \|\boldsymbol{k}\|^{-\rho}$
- $E(\omega) = \int dx\, \left\|\widehat{v'}(x,\omega)\right\|^2 = \int dk\, \left\|\widehat{\widehat{v'}}(k,\omega)\right\|^2 \sim \text{cst.}\, \omega^{d-1}$

Simple noise $O_t(x)$: v' fractal in x
Usual transport noise

Gaussian noise

Noise $= dN_t(x) = dB_t(x)$

$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sigma(x) \circ \frac{dN_t}{dt} = \nabla \times \left(\psi *\circ \frac{dN_t}{dt}\right) = \sum_k ik \times \sqrt{\Gamma_1(k)}\, e^{ik\cdot x} \circ \frac{d\widehat{N_t}(k)}{dt}$$

increments of Q-Wiener process
(Gaussian, divergence-free, white wrt t)

Gaussian noise

$$\mathbb{E}\{\, dB_t(x)\, dB_t(x+\delta x)\} \propto \delta(\delta x)\, dt$$

Resseguier et al. 2017b, 2020, 2021
For transport noise

Full turbulence field

$$\frac{1}{dt}\mathbb{E}\left\|\widehat{dB_t}(k)\right\|^2 = 1$$

$$\mathbb{E}\left\|\widehat{v'}(k)\right\|^2 = \frac{1}{dt}\|\boldsymbol{k}\|\Gamma_1 \propto \left(1 + \frac{\|k\|^2}{k_0^2}\right)^{\frac{-\rho-d+1}{2}}$$

- $E(\|k\|) = \oint d\theta_k \|k\| \left\|\widehat{v'}(k)\right\|^2 \sim \text{cst.}\, \|\boldsymbol{k}\|^{-\rho}$
- $E(\omega) = \int dx\, \left\|\widehat{v'}(x,\omega)\right\|^2 = \int dk\, \left\|\widehat{\widehat{v'}}(k,\omega)\right\|^2 \sim \text{cst.}\, \omega^{d-1}$

Not realistic

Better noise $O_t(x)$: v' fractal in x and t
Causal simulation with scale-dependent Ornstein Uhlenbeck

$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sigma(x) \circ \frac{dN_t}{dt}$$
$$= \nabla \times \left(\psi *\circ \frac{dN_t}{dt} \right)$$
$$= \sum_k ik \times \sqrt{\Gamma_1(k)}\, e^{ik\cdot x} \circ \frac{\widehat{dN_t}(k)}{dt}$$

~~increments of Q-Wiener process~~
(Gaussian, divergence-free, ~~white wrt t~~)

Gaussian noise

Noise $= dN_t(x) = O_t(x)\,dt$

Gaussian noise
$O_t \sim \mathcal{GP}(0, \delta)$
$\mathbb{E}\{ O_t(x)\, O_t(x+\delta x)\} \propto \delta(\delta x)$

Spectrum

$$\mathbb{E}\left\|\hat{\hat{O}}(k,\omega)\right\|^2 \propto \frac{\tau_O(k)}{1+(\tau_0(k)\omega)^2} \quad \text{with } \tau_O(k) \propto \left(1 + \frac{\|k\|^2}{k_0^2}\right)^{-\frac{1}{2}\frac{\rho-1}{\gamma-1}}$$

Causal simulation

$$d\,\widehat{O}_t(k) = -\frac{1}{\tau_0(k)} \widehat{O}_t(k)\,dt + \sqrt{\frac{2}{\tau_0(k)}}\,\widehat{dW_t^1}(k)$$

$O_{t=0}$ ~ space white noise
dW_t^1 ~ space-time white noise

Avellaneda & Majda (1990,1992),
Piterbarg & Ostrovskii (1997);
Chaves et al. (2003),
Reneuve & chevillard (2020)
Chatelain et al. (2025)
$H_{eul} = \frac{1}{2}(\rho - 1)$
$\beta = \frac{z}{2} = \frac{1}{2}\frac{\rho-1}{\gamma-1} = \frac{H_{eul}}{\gamma-1}$

Cifani & Flandoli (2025)
Blessing, Crisan, & Lang (2026+)
For transport noise

Better noise $O_t(x)$: v' fractal in x and t

Gaussian noise

Noise $= dN_t(x) = O_t(x)\,dt$

Gaussian noise

$O_t \sim \mathcal{GP}(0,\delta)$

$\mathbb{E}\{\,O_t(x)\,O_t(x+\delta x)\} \propto \delta(\delta x)$

Avellaneda & Majda (1990,1992),
Piterbarg & Ostrovskii (1997);
Chaves et al. (2003),
Reneuve & chevillard (2020)
Chatelain et al. (2025)

$H_{eul} = \frac{1}{2}(\rho - 1)$

$\beta = \frac{z}{2} = \frac{1}{2}\frac{\rho-1}{\gamma-1} = \frac{H_{eul}}{\gamma-1}$

Cifani & Flandoli (2025)
Blessing, Crisan, & Lang (2026+)

For transport noise

$v = u + v'$

Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sigma(x) \circ \frac{dN_t}{dt}$$
$$= \nabla \times \left(\psi *\circ \frac{dN_t}{dt}\right)$$
$$= \sum_k ik \times \sqrt{\Gamma_1(k)}\, e^{ik\cdot x} \circ \frac{d\widehat{N}_t(k)}{dt}$$

~~increments of Q-Wiener process~~
(Gaussian, divergence-free, ~~white wrt t~~)

Full turbulence field

$$\mathbb{E}\left\|\widehat{\widehat{v}}'(k,\omega)\right\|^2 \propto \frac{\tau_O(k)}{1+(\tau_O(k)\omega)^2}\,\mathbb{E}\left\|\widehat{v}'(k,t=0)\right\|^2 \text{ with } \mathbb{E}\left\|\widehat{v}'(k,t=0)\right\|^2 = \left(1+\frac{\|k\|^2}{k_0^2}\right)^{\frac{-\rho-d+1}{2}}$$

and $\tau_O(k) \propto \left(1+\frac{\|k\|^2}{k_0^2}\right)^{-\frac{1}{2}\frac{\rho-1}{\gamma-1}}$

- $E(\|k\|) = \oint d\theta_k \|k\| \int dt \left\|\widehat{v}'(k,t)\right\|^2 = \oint d\theta_k \|k\| \int d\omega \left\|\widehat{\widehat{v}}'(k,\omega)\right\|^2 \sim \text{cst.}\,\|\boldsymbol{k}\|^{-\rho}$
- $E(\omega) = \int dx \left\|\widehat{v}'(x,\omega)\right\|^2 = \int dk \left\|\widehat{\widehat{v}}'(k,\omega)\right\|^2 \sim \text{cst.}\,\omega^{-\gamma}$ Realistic

similar to fractional Brownian motion

Better noise $O_t(x)$: v' fractal in x **and t**

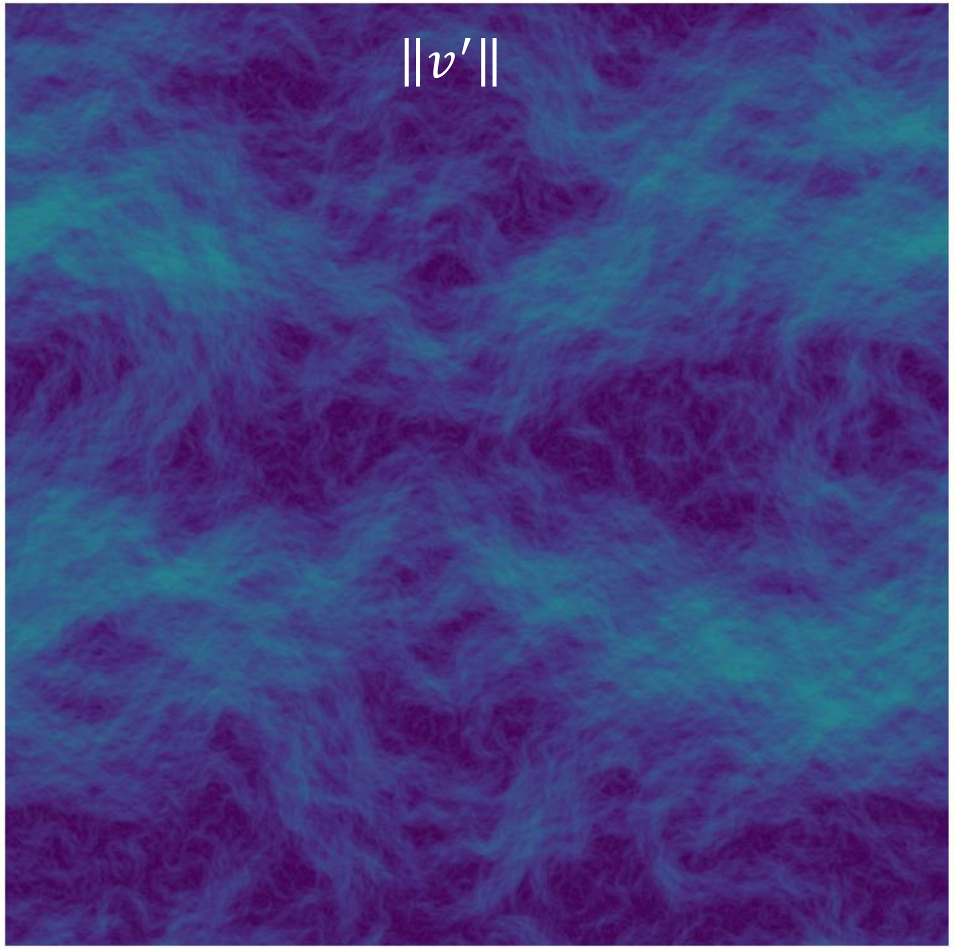
$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sigma(x) \circ \frac{\boldsymbol{dN_t}}{\boldsymbol{dt}}$$
$$= \nabla \times \left(\psi *\circ \frac{\boldsymbol{dN_t}}{\boldsymbol{dt}}\right)$$
$$= \sum_k ik \times \sqrt{\Gamma_1(k)}\, e^{ik\cdot x} \circ \frac{\widehat{\boldsymbol{dN_t}}(\boldsymbol{k})}{\boldsymbol{dt}}$$

~~increments of Q-Wiener process~~
(Gaussian, divergence-free, ~~white wrt t~~)



$dN_t(x) = O_t(x,t)dt$

$$d\,\widehat{O}_t(k) = -\frac{1}{\tau_0(k)}\widehat{O}_t(k)\,dt + \sqrt{\frac{2}{\tau_0(k)}}\,\widehat{dW_t^1}(k)$$

$O_{t=0}$ ~ space white noise
dW_t^1 ~ space-time white noise

$$\tau_O(k) \propto \left(1 + \frac{\|k\|^2}{k_0^2}\right)^{-\frac{1}{2}\frac{\rho-1}{\gamma-1}}$$

- $E(\|k\|) \sim \text{cst.}\, \|\boldsymbol{k}\|^{-\rho}$
- $E(\omega) \sim \text{cst.}\, \omega^{-\gamma}$

Even better noise $O_t(x)$: v' multifractal in x and t
Gaussian Multiplicative Chaos (GMC)

Peirera et al. (2016)
Chevillard et al. (2019, 2020)
Ruffenach & Chevillard (2026)

Intermittency → $\epsilon_t(x)$; Gaussian noise → $O_t(x)$

$$\text{Noise} = dN_t(x) = \epsilon_t(x) O_t(x) dt$$

Gaussian noise

$$O_t \sim \mathcal{GP}(0, \delta)$$
$$\mathbb{E}\{ O_t(x)\, O_t(x + \delta x)\} \propto \delta(\delta x)$$

Intermittency

$$\epsilon_t(x) = \exp\left(\log \tilde{\epsilon}_t(x) - Var(\log \tilde{\epsilon}_t(x))\right)$$
$$\log \tilde{\epsilon}_t \sim \mathcal{GP}(0, \log_+ \ )$$
$$\mathbb{E}\left\|\widehat{\log \tilde{\epsilon}_t}(k)\right\|^2 = \boldsymbol{c_2} \frac{k_0}{\|k\|}$$

$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sigma(x) \circ \frac{\boldsymbol{dN_t}}{\boldsymbol{dt}}$$
$$= \nabla \times \left(\psi * \circ \frac{\boldsymbol{dN_t}}{\boldsymbol{dt}}\right)$$
$$= \sum_k ik \times \sqrt{\Gamma_1(k)}\, e^{ik\cdot x} \circ \frac{\widehat{\boldsymbol{dN_t}}(\boldsymbol{k})}{\boldsymbol{dt}}$$

~~increments of Q-Wiener process~~
(Gaussian, divergence-free, ~~white wrt t~~)

Even better noise $O_t(x)$: v' multifractal in x **and t**
Gaussian Multiplicative Chaos (GMC)

Peirera et al. (2016)
Chevillard et al. (2019, 2020)
Ruffenach & Chevillard (2026)

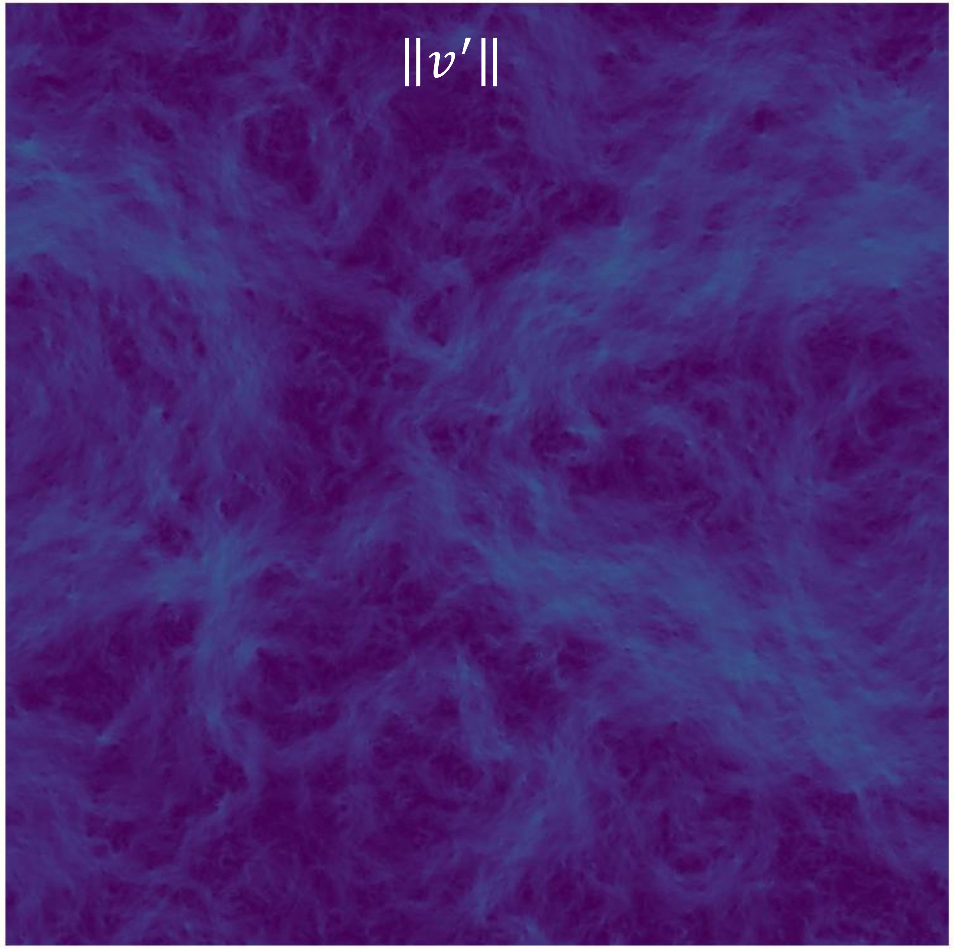
$$v = u + v'$$

Resolved fluid velocity:
u

Unresolved fluid velocity:

$$v'(x,t) = \sigma(x) \circ \frac{dN_t}{dt}$$
$$= \nabla \times \left(\psi *\circ \frac{dN_t}{dt}\right)$$
$$= \sum_k ik \times \sqrt{\Gamma_1(k)}\, e^{ik\cdot x} \circ \frac{d\widehat{N}_t(k)}{dt}$$

~~increments of Q-Wiener process~~
(Gaussian, divergence-free, ~~white wrt t~~)



Noise $= dN_t(x) = \epsilon_t(x) O_t(x) dt$

$$d\,\widehat{O}_t(k) = -\frac{1}{\tau_O(k)} \widehat{O}_t(k)\, dt + \sqrt{\frac{2}{\tau_O(k)}}\, \widehat{dW_t^1}(k)$$

$O_{t=0}$ ~ space white noise
dW_t^1 ~ space-time white noise

$$\tau_O(k) \propto \left(1 + \frac{\|k\|^2}{k_0^2}\right)^{-\frac{1}{2}\frac{\rho-1}{\gamma-1}}$$

Similar construction for $\log \tilde{\epsilon}_t$

- $E(\|k\|) \sim \text{cst.}\, \|\boldsymbol{k}\|^{-\rho}$
- $E(\omega) \sim \text{cst.}\, \omega^{-\gamma}$

Open questions

- **Mathematical meaning** of the transport noise SPDE : $d_t u + (u dt + \sigma \circ dN_t) \cdot \nabla u = dF$
  - For $\sigma \circ dN_t$ fractal in space and time (rough path?)
  - For $\sigma \circ dN_t$ Gaussian multiplicative chaos with steady intermittency
  - For $\sigma \circ dN_t$ Gaussian multiplicative chaos with unsteady intermittency
- **Wellposedness** of the transport noise SPDE
- **Filtering / Bayesian inverse problem**
  - With Fractal
  - With Gaussian multiplicative chaos
  - With transport noise SPDE

Conclusion

Conclusion

- **Transport noise**
  - Established framework
    - Solid theory
    - Many codes and calibration methods
    - A full community
  - Still new research paths appearing
    - Compressible flows, waves
    - Eulerian-Lagrangian, point vortex methods
    - Girsanov-based maximum likelihood estimation
    - Link to free multiplicative Brownian motion
- **Multiplicative-noise-based generative artificial intelligence**
  - New research domain
  - Diffusion, Langevin, flow matching, …
  - So many open questions
- **Beyond white noise**
  - Fractal transport noise : some first results
  - Multifractal transport noise : *terra incognita*