Stochastic Partial Differential Equations in Fluid Dynamics

Dan Crisan

Imperial College London

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Talk Synopsis

- Motivation: Uncertainty Quantification for GFD models
- Examples
  - 2D Euler Equation with Transport Noise
  - Rotating Shallow Water Equation with SALT Noise
  - Fluid Dynamics Models driven by Fractional Brownian Motion
- Stochastic Transport in Upper Ocean Dynamics (STUOD)
- Open problems
  - Set 1: Analysis and derivation
  - Set 2: Long-time behaviour and computation
  - Set 3: Data-driven and machine-learning approaches
- Final Remarks

Stochastic Partial Differential Equations are important !

- Despite major advances, fundamental open problems remain at the interface of:
  - mathematical analysis,
  - numerical approximation,
  - and real-world applications.
- These questions matter both for theory and for the reliability of stochastic models used in practice.

Motivation: Uncertainty Quantification for Geophysical Fluid Dynamics models

- Numerical weather and climate modeling is based on the discretization of the continuous equations of motion.
- Such models can be characterized in terms of their dynamical core, which describes the resolved scales of motion, and the physical parameterizations, which provide estimates of the subgrid-scale effect of processes that cannot be resolved.
- This general approach has been successful in that skillful predictions of weather and climate are now routinely made.
- However, it has become apparent that current state-of-the-art models exhibit persistent and systematic shortcomings due to an inadequate representation of unresolved processes.

- Despite the continuing increase of computing power, which allows numerical weather and climate prediction models to be run with ever-higher resolution, the multiscale nature of GFD implies that many important physical processes (e.g., tropical convection, gravity wave drag, and microphysical processes) are still not resolved.
- Parameterizations of subgrid-scale processes contain closure assumptions and related parameters with inherent uncertainties. fluctuations, triggering noise-induced regime transitions, capturing responses to changes in the external forcing.

Stochastic parameterizations - empirically derived or based on rigorous mathematical and statistical concepts have great potential to increase the predictive capability of next-generation weather and climate models.

J. Berner, U. Achatz, L. Batté, L. Bengtsson, A. de la Camara, H. M. Christensen, M. Colangeli, D. R. B. Coleman, D. Crommelin, S. I. Dolaptchiev, C. L. E. Franzke, P. Friederichs, P. Imkeller, H. Järvinen, S. Juricke, V. Kitsios, F. Lott, V. Lucarini, S. Mahajan, T. N. Palmer, C. Penland, M. Sakradzija, J. von Storch, A. Weisheimer, M. Weniger, P. D. Williams, and J. Yan, STOCHASTIC PARAMETERIZATION, Toward a New View of Weather and Climate Models, AMERICAN METEOROLOGICAL SOCIETY, 2017.

Incorporating stochasticity into fluid dynamics models is essential for capturing the inherent *uncertainty, variability, and complexity* of real-world fluid flows.

Deterministic fluid dynamics models often overlook the influence of *unresolved small scales*, *measurement noise*, and *chaotic behavior*, particularly in turbulent regimes.

Stochastic models address these limitations by:

- Representing **subgrid-scale effects**
- Enhancing **predictive robustness**
- Enabling **data assimilation** and **uncertainty quantification**

This approach is especially valuable in applications such as *weather prediction*, *ocean modeling*, and *climate science*, where probabilistic forecasts and model realism are critical.

High versus Low Res simulation: Euler equation with forcing and damping

Deterministic versus Stochastic models: Stochastic Euler

Numerics done by Wei Pan (ECMWF)

Two-dimensional stochastic Euler vorticity equation

$$d\omega_t + \mathbf{u}_t \cdot \nabla \omega_t dt + \mathcal{A}(\omega_t) dW_t = 0$$

where

- $\mathbf{u}$ velocity field
- ω vorticity field
- W Brownian motion
- $\mathcal{A}(\omega_t)$ amplitude of the stochastic term

To choose $\mathcal{A}(\omega_t)$ we follow the fluid particles/Lagrangian fluid parcels:

Evolution of Lagrangian fluid parcels:

$$d_t \mathbf{x}_t = \mathbf{u}_t(\mathbf{x}_t) \qquad \Rightarrow d_t \mathbf{x}_t = \mathbf{u}_t(\mathbf{x}_t) dt + \xi(\mathbf{x}_t) \circ dW(t)$$

leading to

$$d\omega_t + \mathbf{u}_t \cdot \nabla \omega_t dt + (\xi \cdot \nabla \omega_t) \circ dW_t = 0.$$

Theorem

If $\omega_0 \in \mathcal{W}^{k,2}(\mathbb{T}^2)$ *is a divergence free function then the two-dimensional stochastic Euler vorticity equation*

$$d\omega_t + u_t \cdot \nabla \omega_t dt + \sum_{i=1}^{\infty} (\xi_i \cdot \nabla \omega_t) \circ dW_t^i = 0$$

admits a global $\mathcal{F}_t$*-adapted solution* $\omega = \{\omega_t, t \in [0,\infty)\}$ *in the space* $C\left([0,\infty); \mathcal{W}^{k,2}(\mathbb{T}^2)\right)$. *In particular, if* $k \geq 4$ *the solution is classical. Moreover if* $\tilde{\omega} = \{\tilde{\omega}_t, t \in [0,\infty)\}$ *is another solution of the equation, then, for all* $T \in [0,\infty)$ *there exists a positive constant* C *independent of the two solutions such that*

$$\mathbb{E}[e^{-CA_t}||\omega_t - \tilde{\omega}_t||_{k-1,2}^2] \leq ||\omega_0 - \tilde{\omega}_0||_{k-1,2}^2 \tag{1}$$

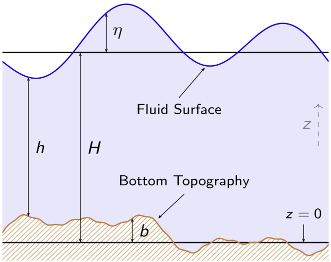
In (1) A*, is the process defined as* $A_t := \int_0^t (||\omega_s||_{k,2} + 1) ds$*, for any* $t \geq 0$*. In particular the solution of the equation is unique.*

O. Lang, D. Crisan, Well-posedness for a stochastic 2D Euler equation with transport noise. Stoch PDE: Anal Comp 11, 433–480 (2023).

The Stochastic Rotating Shallow Water Equation with SALT Noise

The deterministic model

$$\frac{D}{Dt}u_t + f\hat{z} \times u_t + g\nabla h_t = 0$$
$$\frac{\partial h_t}{\partial t} + \nabla \cdot (h_t u_t) = 0$$



h: height of the fluid column

H: average height of fluid column over domain

η: elevation of fluid surface relative to H

b: bottom topography

$H + \eta = h + b$: z-coordinate of fluid surface

The inviscid model:

$$\frac{D}{Dt}u_t + f\hat{z} \times u_t + g\nabla h_t = 0 \tag{2a}$$

$$\frac{\partial h_t}{\partial t} + \nabla \cdot (h_t u_t) = 0 \tag{2b}$$

- $\frac{D}{Dt} := \frac{\partial}{\partial t} + u \cdot \nabla$ is the material derivative.
- $u = (u^1, u^2)$ is the horizontal fluid velocity vector field
- h is the height of the fluid column
- f is the Coriolis parameter, $f = 2\Theta \sin\varphi$ where Θ is the rotation rate of the Earth and φ is the latitude; $f\hat{z} \times u = (-fu^2, fu^1)$, where $\hat{z}$ is a unit vector pointing away form the centre of the Earth
- g is the gravitational acceleration

We can formally re-write a viscous version of the RSW system: $X := (u, h)$ and then

$$d_t X_t + F(X_t) = 0 \tag{3}$$

where $F(X_t)$ denotes

$$F\begin{pmatrix} u \\ h \end{pmatrix} = \begin{pmatrix} u \cdot \nabla u + f\hat{z} \times u + g\nabla h - \nu\Delta u \\ \nabla \cdot (hu) - \eta\Delta h \end{pmatrix}. \tag{4}$$

Theorem

If $(u_0, h_0) \in (\mathcal{W}^{1,2}(\mathbb{T}^2))^3$, then the SPDE

$$du_t + \left[u_t \cdot \nabla u_t + f\hat{z} \times u_t + g\nabla h_t\right] dt + \sum_{i=1}^{\infty} \left[(\mathcal{L}_i + \mathcal{A}_i)u_t\right] \circ dW_t^i = \nu \Delta u_t dt \tag{5a}$$

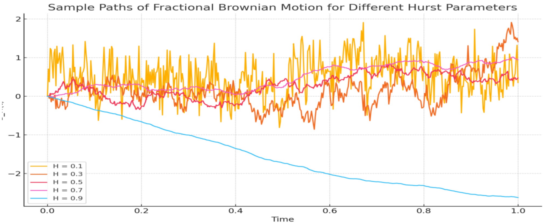
$$dh_t + \nabla \cdot (h_t u_t) dt + \sum_{i=1}^{\infty} \left[\nabla \cdot (\xi_i h_t)\right] \circ dW_t^i = \eta \Delta h_t dt, \tag{5b}$$

where ξ_i are divergence-free vector fields, $\mathcal{L}_i u := \xi_i \cdot \nabla u$, $\mathcal{A}_i u := u_j \nabla \xi_i^j$ and W^i are independent Brownian motions admits a unique maximal strong solution.

D. Crisan, O. Lang, Well-Posedness Properties for a Stochastic Rotating Shallow Water Model, J Dyn Diff Equations 36, 3175–3205 (2024).

- Most stochastic models use standard Brownian motion for incorporating stochasticity into the dynamics. Real-world turbulence often displays long-range dependence and memory.
- Fractional Brownian motion provides a richer class of stochastic perturbations: it allows to account for temporal correlations.
- Eddies can influence each other across scales and times and fractional Brownian motion better captures this. The use of fractional Brownian motion opens the door to modelling more realistic fluid behaviours.

- fBm $W^H(t)$ with Hurst parameter $H \in (0,1)$ is a Gaussian process with $W^H(0) = 0$, $\mathbb{E}[W^H(t)] = 0$ and covariance $\mathrm{Cov}(W^H(t), W^H(s)) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$.
  Sample paths are α-Hölder continuous for any $\alpha < H$.



- Non-Markovian, with Hurst parameter H controlling the time-correlation.
- **When $H > 0.5$:** Positive correlation between distant increments.
- **When $H < 0.5$:** Negative correlation between distant increments.

- **Time correlation of increments:**
  - Let $\Delta W^H(s) = W^H(s+1) - W^H(s)$
  - Correlation between increments at times s and t:

$$\rho_H(t-s) = \rho_H(s,t) = \mathbb{E}[\Delta W^H(s)\Delta W^H(t)] = \frac{1}{2}\left(|t-s+1|^{2H} + |t-s-1|^{2H} - 2|t-s|^{2H}\right)$$

- **Special case:** When $H = \frac{1}{2}$, fBm reduces to standard Brownian motion (with independent increments)

- The correlation between increments at times s and t, with $\tau = t - s$ has the following asymptotic decay for large τ: $\rho_H(\tau) \sim H(2H-1) \cdot \tau^{2H-2}$
  - Slow decay for $H > 0.5$: long memory.
  - Fast decay for $H < 0.5$: short memory.

- Define the function spaces:

$$V^T := C(0,T;\mathcal{B}_\alpha) \cap C^\gamma(0,T;\mathcal{B}_{\alpha-\gamma})$$

where $\mathcal{B}_\alpha = W^{2\alpha,2}$,

$$y \in V^T, \quad ||y||_{V^T} = \max\left\{||y||_{0,\alpha}, ||y||_{\gamma,\alpha-\gamma}\right\}$$

- For $y \in C([0,T],\mathcal{B}_\alpha)$, we have that

$$||y||_{0,\alpha} = \sup_{t\in[0,T]} ||y_s||_\alpha \qquad ||x||_\alpha := ||x||_{W^{2\alpha,2}(\mathbb{T}^2)}$$

- For $y \in C^\gamma([0,T],\mathcal{B}_\beta)$, we have that

$$||y||_{\gamma,\beta} = \sup_{s,t\in[0,T]} \frac{||y_t - y_s||_\beta}{|t-s|^\gamma}.$$

General equation

$$d\omega_t + (\mathcal{D} + \mathcal{E})\,\omega_t dt + \mathcal{F}\omega_t dW_t^H = 0 \tag{6}$$

where

- $\mathcal{D} : \mathcal{B}_\alpha \to \mathcal{B}_{\alpha-\beta}$ is a nonlinear operator,
- $\mathcal{F} : \mathcal{B}_\alpha \to \mathcal{B}_{\alpha-\beta}$ is a (possibly) nonlinear operator,
- $\mathcal{E} : \mathcal{B}_1 \to \mathcal{B}_1$ is an (unbounded) linear operator with domain $D\,(\mathcal{B}_1)$ dense in $\mathcal{B}_1$ for which there exists an associated semigroup $S_t : \mathcal{B}_1 \to \mathcal{B}_1$

$$\frac{d}{dt} S_t f = \mathcal{E} f.$$

- W^H is a fBm with $H > 1/2$

Theorem

Under assumptions, equation (6) admits a unique local mild solution in V^T.

Corollary

Under sufficient smootheness assumptions, the solution of equation (6) admits a unique local strong solution.

A particular case: Incompressible viscous model

Consider the 2D vorticity equation on the torus $\mathbb{T}^2$:

$$d\omega_t + (u_t \cdot \nabla\omega_t - \Delta\omega_t)dt + \xi \cdot \nabla\omega_t dW_t^H = 0 \tag{7}$$

where:

- ω is the fluid vorticity,
- $u = K * \omega$ is the velocity field (Biot-Savart law),
- W^H is a fBm with $H > 1/2$.
- $\int_{\mathbb{T}^2} \omega_t(x)dx = 0$
- $\xi \in \mathbb{B}^{\alpha+1}$, $\nabla \cdot \xi = 0$.

- We can also have a countable set of fBms with the common/different parameter H.

Consider the mild form of equation (7):

$$\omega_t = S_t\omega_0 - \int_0^t S_{t-s}(u_s \cdot \nabla\omega_s)ds - \int_0^t S_{t-s}(\xi \cdot \nabla\omega_s)dW_s^H \tag{8}$$

Theorem 1

Equation (8) has a unique local solution in the functional space

$$C(0,T;\mathcal{B}_\alpha) \cap C^\gamma(0,T;\mathcal{B}_{\alpha-\gamma}).$$

Estimating the Hurst parameter H

- We show that H can be estimated for an (infinite dimensional) SPDE, with non-constant diffusion coefficient.
- The drift coefficient can be quite general, as long as it is bounded in time.
- We use observed trajectories of $\langle \omega_t, \varphi \rangle$, where φ is a suitably chosen test function.
- This generalizes previous work, e.g., Tudor et. al: 1-dim SDEs, driven by fBm (no diffusion term).

Theorem (Hurst parameter estimation)

Define

$$H_n := \frac{1}{2}\left(\frac{1}{\log 2} \log \frac{\sum_{\{j, [\frac{j}{2^n}, \frac{j+1}{2^n}] \subset [0,t]\}} \left(\langle \omega_{t^n_{j+1}}, \varphi \rangle - \langle \omega_{t^n_j}, \varphi \rangle \right)^2}{\sum_{\{j, [\frac{j}{2^{n+1}}, \frac{j+1}{2^{n+1}}] \subset [0,t]\}} \left(\langle \omega_{t^{n+1}_{j+1}}, \varphi \rangle - \langle \omega_{t^{n+1}_j}, \varphi \rangle \right)^2} - 1 \right)$$

Then $\mathbb{P}$-almost surely $\lim_{n\to\infty} H_n = H$.

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Inria

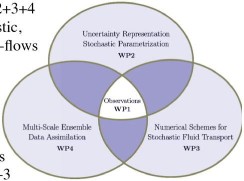
Stochastic Transport in Upper Ocean Dynamics

Bertrand Chapron
Dan Crisan
Darryl Holm
Etienne Mémin

Building new capabilities

Stochastic Transport in Upper Ocean Dynamics

- Decipher and forecast upper oceanic dynamics WP1+2+3+4
- Deliver a complete stochastic, data-driven theory for 3D-flows WP1+2+3+4
- Derive stochastic dynamics of the surface layer WP2+3
- Develop satellite and in situ data analysis techniques to calibrate our stochastic model WP1+2



Theoretical Analysis of SPDEs

- Models analysed:
  - 2D/3D Euler and Navier–Stokes equations
  - Great Lake equation
  - Rotating Shallow Water equation
  - Thermal Quasi–Geostrophic equation
  - Coupled Ocean–Atmosphere equations
  - SPDEs for viscous fluids equations
  - SPDEs on bounded domains
- Blow-up criteria/ global solutions/Regularity results: Thermal Quasi-Geostrophic Equation with Stochastic Lie Transport, 2D Euler with transport noise/ Rotating Shallow Water with LU noise, 2D Navier-Stokes Equation with SALT noise

- Dependence on the initial condition/the diffusion coefficients for 2D Euler Equation with SALT noise.
- Approximation rates/Consistency results for the corresponding numerical schemes for the thermal Quasi-Geostrophic Equation with SALT noise.
- Rigorous characterization of GFD models as critical points of stochastic/rough action functionals.
- The explicit description of the pressure term in incompressible models.
- Global solutions for stochastically controlled models.
  - Viscous compressible and incompressible models, e.g. incompressible 3D Navier–Stokes equation in vorticity form, viscous 2D rotating shallow water.
  - Inviscid compressible models, e.g. Burgers' equation on the 1D/2D torus, 2D inviscid rotating shallow water.
  - Inviscid incompressible models, e.g. incompressible Euler equation in vorticity form on the 3D torus.

Joint work with: Oana Lang, Romeo Mensah, Darryl Holm, Etienne Memin, James Leahy, Wei Pan, Torstein Nilssen, Daniel Goodair, Filippo Giovagnini.

Data Assimilation & Calibration of SPDEs

- Model Reduction: coarsen the grid used for the numerical algorithm (the evolution of the particles) that approximates the dynamical system. Use SALT-LU models to account for the small/fast scales.
- We have developed methodologies for calibrating the stochasticity in SALT-LU models.
  - Lagrangian Calibration: Euler, 2 layer QG, TQG (synthetic data)
  - Eulerian Calibration: general principle, RSW (synthetic data, work in progress for real data)
  - Deep Learning
  - Generative models: RSW (synthetic data)
- Reduced order models for LU/SALT parametrization
  - Comparison of Stochastic Parametrization Schemes using Data Assimilation on Triad Models
  - Helicity-preserving stochastic triads and Energy preserving stochastic triad.

- We have adapted particle filters to work in high dimensions.
- The methodology has already been tested on several benchmark models.

Add-on techniques:

- Nudging
- Jittering
- Localization
- Tempering

$$\pi_{t-1}^N \overbrace{\longrightarrow\longrightarrow}^{\text{nudging}} p_t^N \quad \overbrace{\longrightarrow\longrightarrow\longrightarrow\longrightarrow}^{\text{localization+tempering+jiterring}} \quad \pi_t^N$$

Test Cases

- 2D Euler with forcing and damping
- 2-layer Quasi-Geostrophic model
- Rotating Shallow Water model
- Thermal Quasi-Geostrophic
- 3-layer Quasi-Geostrophic model
- Camassa-Holm equation (parallel PF)

Joint work with: Joaquin Miguez, Oana Lang, Etienne Memin, Roland Potthast, Peter Jan van Leeuwen, Erwin Luesink, Wei Pan, Alexander Lobbe.

OP Set 1: Analysis and derivation

- **Well-posedness and regularity for nonlinear SPDEs driven by rough noise**
  - Many physically relevant models remain analytically challenging.
  - Open questions include global existence, uniqueness, and regularity.
  - Long-time behaviour is often poorly understood, even when local well-posedness is known.

- **Rigorous derivation of SPDEs as effective multiscale models**
  - Random forcing is often introduced to represent unresolved small scales.
  - Deriving correct stochastic terms from first principles remains difficult.
  - Convergence from microscopic deterministic dynamics to macroscopic SPDEs is still incomplete in many regimes.

OP Set 2: Long-time behaviour and computation

- **Long-time behaviour and statistical properties**
  - Existence and uniqueness of invariant measures can already be delicate.
  - Quantifying ergodicity, mixing rates, metastability, and large deviations is even harder.
  - These issues are central in climate applications such as tipping points and regime transitions.

- **Numerical approximation of SPDEs**
  - Deterministic discretization methods often fail to preserve stochastic structure.
  - Stable and structure-preserving schemes are still under active development.
  - Strong and weak error analysis is incomplete for many nonlinear models.

OP Set 3: Data, learning, and prediction

- **Data-driven and machine-learning approaches**
  - How can stochastic forcing be identified from partial observations?
  - How should uncertainty be quantified in learned models?
  - How can physics-based SPDEs and data-driven corrections be combined while preserving stability and interpretability?

- **Using stochastic modeling to improve predictive skill**
  - Stochastic terms are often introduced to represent uncertainty.
  - Mathematical criteria for optimal stochastic parameterization are not yet well understood.
  - Principled frameworks for model reduction and uncertainty representation remain a major challenge.

- Stochastic PDEs provide a powerful framework for modelling uncertainty and multiscale effects in fluid dynamics.
- They can bridge the gap between rigorous mathematical theory and real-world applications, particularly in climate and geophysical flows.
- Significant progress has been made in:
  - Well-posedness and regularity theory
  - Stochastic fluid models (Euler, shallow water, etc.)
  - Incorporation of structured noise (e.g.Brownian noise, fractional noise)
- However, major challenges remain:
  - Long-time behaviour and statistical properties
  - Structure-preserving numerical methods
  - Rigorous multiscale derivation of stochastic models
  - Data-driven modelling and uncertainty quantification
- Advances in these areas will be crucial for applications in climate science, biology, engineering, and beyond.