

On a “dynamic” approximation of a Lévy process and its application to Lévy-driven SDEs

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Plan:

- General knowledge
- “Dynamic” approximation of a Lévy process
- Optimal cutting level
- Euler scheme for a Lévy-driven SDE
- Simulations

Introduction

Z is a Lévy process: independent increments, time-homogeneous.

Lévy triplet: (ℓ, Q, ν) .

Lévy-Ito decomposition:

$$Z(t) = Z(0) + \ell t + QB(t) + \int_0^t \int_{0 < |y| \leq 1} y \tilde{N}(dy, ds) + \int_0^t \int_{|y| > 1} y N(dy, ds)$$

where

$$N((0, t], A) := \#\{s \in (0, t] : \Delta Z_s \in A\}, \quad A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$$

is the Poisson random measure associated with Z , ν is the Lévy measure,
 $\tilde{N}(ds, dz) = N(ds, dz) - \nu(dz)ds$ is the compensated Poisson random measure.

Problem for simulations:

Z : countably many jumps in $[0, T]$.

- **How to simulate Z_t ?**
- **How to simulate a Lévy-type process? Lévy-driven SDE?**

How to simulate an inf. divisible distribution:

$$Z_t = \sum_{k=1}^n (Z_{t_k} - Z_{t_{k-1}}) \doteq \sum_{k=1}^{\infty} Z_{t_k - t_{k-1}}, \quad t_k \in \Pi_n$$

- Indirect way: use the construction of Z
e.g., Chambers, Mallow, Struck'76 (for α -stable), Michael, Schucany, Haas'76 (NIG)
- Use a distribution which is easy to generate, then apply Acceptance-Rejection method
e.g., for Gamma distribution, Meixner distribution etc.
- Use a Compound Poisson approximation: e.g. Damien, Laud and Smith'95, Bondesson'82.
- Use Compound Poisson, but substitute small jumps by normal variables
Asmussen, Rosiński'01

$$Z_{\delta}^{(1)(2),AR}(t) := t\left(\ell - \int_{\delta < |u| \leq 1} u\nu(du)\right) + \textcolor{red}{+\sigma(\delta)B(t)} + \sum_{s \leq t} 1_{|\Delta Z(s)| > \delta} \Delta Z(s).$$

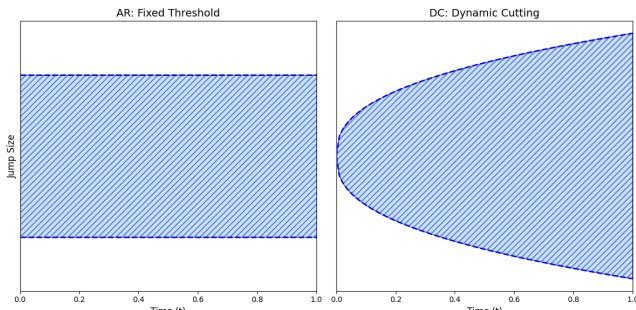
where $\sigma^2(\delta) := \int_{|u| \leq \delta} u^2 \nu(du)$.

“Refined cutting”

- cut away the area in a specific time-dependent way: $|z| \leq \phi(t)$
- why “dynamic cutting”: used in the theoretical construction of the trans. probab. density (Lévy: K., Kulik’13),
- Lévy-type: $(\ell(x), Q(x), \nu(x, du))$

the **parametrix construction** of the trans. prob. dens. of a Lévy-type process
K., Kulik’15, 17, 18,

(further: K., Kulik, Schilling’21, Bogdan, K, Sztonyk’20)



Idea: Given (L, C_0^2) , show that there exists a (Feller) process, associated with (L, C_0^2) .

Idea: $p(t, x, y) = p^0(t, x, y) + R(t, x, y)$, where

a) p^0 is the “proxy” density (small jumps only, less than $\phi(t)$ for some ϕ)

a) $\sup_x \|R(t, x, \cdot)\|_{L_1} \lesssim t^\epsilon$.

Refinement: (e.g., α -stable-like case, $(b(x), 0, \nu(x, du))$,
 $\nu(x, du) = |u|^{-d-\alpha(x)} du$):

$$p(t, x, y) = \tilde{p}(t, x, y) + \tilde{R}(t, x, y),$$

where $\tilde{p} \sim (a(x), 0, b(x), \nu(x, du))$, $\sup_x \|\tilde{R}(t, x, \cdot)\|_{L_1} \lesssim t^\epsilon$.

Problem:

- Simulate $Z(t)$, by cutting away small jumps of size, less than some $\phi(t)$;
- “restore” small jump part.

Setting and Assumptions (symmetric case)

$$N(r) := \nu(u > 0 : u > r), \quad \tau(t) := \sup\{r \geq 0 : N(r) \geq 1/t\}, \quad t, r > 0. \quad (1)$$

Pruitt function (Pruitt'81):

$$\psi^L(\xi) := \int_{|u\xi| \leq 1} |u\xi|^2 \nu(du).$$

Assumptions:

ν is abs. continuous; (A1)

$N(r) \asymp \psi^L(1/r), \quad r > 0.$ (A2)

From (A2): $\exists \alpha_0, \alpha \in (0, 2)$ s.t.

$$r^{-\alpha_0} \lesssim N(r) \lesssim r^{-\alpha}, \quad r \in (0, 1]$$

First-and-second order approximations

Cutting level: $\tau((sh)^\epsilon)$, $\epsilon, h > 0$

First order approximation:

$$Z_{h,\epsilon}^{1,DC}(t) := Z_0 + \ell t + Z_{h,\epsilon}^{CP,+}(t) + Z_{h,\epsilon}^{CP,-}(t), \quad t > 0, \quad (2)$$

$$Z_{h,\epsilon}^{CP,+}(t) := \sum_{s \leq t} \Delta Z_s 1_{\Delta Z_s \geq \tau((sh)^\epsilon)}, \quad Z_{h,\epsilon}^{CP,-}(t) := \sum_{s \leq t} \Delta Z_s 1_{\Delta Z_s \leq -\tau((sh)^\epsilon)}$$

are the upward and downward jumps.

Second order approximation: (t –fixed)

$$Z_{h,\epsilon}^{2,DC}(t) := Z_{h,\epsilon}^{1,DC}(t) + \sigma_{h,\epsilon}(t)W, \quad W \sim N(0, 1) \quad (3)$$

$$V^{(2)}(h) := \frac{1}{h} \int_0^h \int_{|u| \leq \tau(s^\epsilon)} u^2 \nu(du) ds, \quad \sigma_{h,\epsilon}^2(t) := tV^{(2)}(th).$$

Weak approx. rates for $f \in C_b^1$: similar to AR'01

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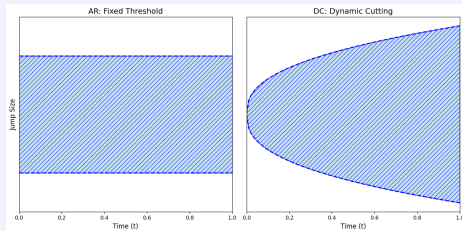


Figure: AR vs DC

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Weak approx. rates for $f \in C_b^1$: similar to AR'01

Weak approximation rates

Theorem 1

Let $f \in C_b^1(\mathbb{R})$. Then

$$\sup_{t \in [0, T]} \left| \mathbb{E}f(Z(t)) - \mathbb{E}f(Z_{h,\varepsilon}^{1,DC}(t)) \right| \leq C^{(1),T} \tau((Th)^\varepsilon)(Th)^{-\varepsilon/2},$$

$$\sup_{t \in [0, T]} \left| \mathbb{E}f(Z(t)) - \mathbb{E}f(Z_{h,\varepsilon}^{2,DC}(t)) \right| \leq C^{(2),T} \tau((Th)^\varepsilon)(|\ln(Th)^\varepsilon| + |\ln T|)$$

Example, α -stable

$$\sup_{t \in [0,1]} \left| \mathbb{E}f(Z(t)) - \mathbb{E}f(Z_{h,\varepsilon}^{1,DC}(t)) \right| \lesssim h^{\frac{\varepsilon}{\alpha} - \frac{\varepsilon}{2}}$$

$$\sup_{t \in [0,1]} \left| \mathbb{E}f(Z(t)) - \mathbb{E}f(Z_{h,\varepsilon}^{2,DC}(t)) \right| \lesssim h^{\frac{\varepsilon}{\alpha}} |\ln h^\varepsilon|.$$

Sketch of the proof

$$R(t) := Z(t) - Z_{h,\varepsilon}^{1,DC}(t) = \int_0^t \int_{B(s,h)} u \tilde{N}(du, ds) = \sum_{j=1}^n Y_j(t/n). \quad (4)$$

$$|\mathbb{E}f(Z(t)) - \mathbb{E}f(Z^{1,DC}(t))| \lesssim \mathbb{E}|R(t)| \leq \sqrt{\mathbb{E}|R(t)|^2} \leq \sqrt{\text{Var } R(t)},$$

$$\begin{aligned} & |\mathbb{E}f(Z^{1,DC}(t) + R(t)) - \mathbb{E}f(Z^{1,DC}(t) + \sigma(t)B(1))| \\ & \leq \int |\mathbb{E}f(x + R(t)) - \mathbb{E}f(x + \sigma(t)B(1))| \mathbb{P}(Z^{1,DC}(t) \in dx) \\ & \lesssim \sigma(t) \int |\mathbb{P}((\sigma(t))^{-1}R(t) < y) - \mathbb{P}(B(1) < y)| dy. \end{aligned}$$

Then Esseen's Thm + Burkholder-Davis-Gundy ineq. for moment est. for $R(t)$.

Optimality

- reasons for DC
- optimal cutting level (e.g., choice of ϵ in $\tau((sh)^\epsilon)$)

Example. $\phi(t)$: cutting level. Define (again, assume symmetry)

$$X_T := \int_0^T \int_{|z| \leq \phi(t)} t^{-\sigma} z \tilde{N}(dt, dz) + \int_0^T \int_{|z| > \phi(t)} t^{-\sigma} z N(dt, dz) =: R_T^\phi + Y_T^\phi,$$

$$\mathcal{E}(\phi) := \text{Var } R_T^\phi, \quad J(\phi) := \mathbb{E} Y_T^\phi.$$

Optimize:

$$\mathcal{E}(\phi) \longrightarrow \min, \quad \text{subject to } J(\phi) = J_0.$$

$$N(r) \lesssim r^{-\alpha}, \quad r \in (0, 1].$$

Proposition

$$(\mathbf{A1})\text{--}(\mathbf{A2}), \quad \sigma \in [0, 1/\alpha), \quad J_0 \geq TN(1) \implies \phi(t) = (c^* t^\sigma) \wedge 1.$$

c^* : from the constraint $J(\phi) = J_0$.

$$\phi(t) = \tau(t^\epsilon) = t^{\epsilon/\alpha} \implies \epsilon = \sigma/\alpha \text{ is optimal.}$$

Lévy-driven SDE

$$X_t = X_0 + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dB_s + \int_0^t \int_{\mathbb{R}} c(s, X_{s-}, z) \tilde{N}(ds, dz).$$

Assumptions

- ❶ A1, A2;
- ❷ Lipschitz + Peano cond. Lip. constants: $L_{a,b}(t)$, $L_c(t, z)$
- ❸ $L_c(t, \cdot) \in L_p$, $p > 1$.
- ❹ $M_p(\cdot) \in L_\zeta([0, T])$ for some $\zeta > 1$,

$$M_p(t) := \begin{cases} \int_{B^c} |L_c(t, z)|^p \nu(dz), & 1 < p < 2, \\ \int_{B^c} |L_c(t, z)|^p \nu(dz) + \left(\int_{B^c} |L_c(t, z)|^2 \nu(dz) \right)^{p/2}, & p \geq 2. \end{cases}$$

- ❺ $\exists \sigma \in (0, 1)$ such that

$$|c(t, x, z)| \lesssim t^{-\sigma}(1 + |x|)|z|, \quad L_c(t, z) \lesssim t^{-\sigma}|z|.$$

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Assumpt

Lip:

① A1, A

② Lipsch

③ $L_c(t,$

④ $M_p(\cdot)$

Peano:

$$|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \leq L_{a,b}(t)|x - y|,$$

$$|c(t, x, z) - c(t, y, z)| \leq L_c(t, z)|x - y|$$

$$|a(t, x) - a(s, x)| + |b(t, x) - b(s, x)| \leq L_{a,b}|t - s|^\gamma(1 + |x|) \quad \gamma < p < 2,$$

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- Combined grid Π_n : uniform grid π_n , mixed with jumps of the ICPP
- Approximate Z using DC (α -stable: $\tau(t) \lesssim t^{1/\alpha}$)

Theorem 2

- a) $p \geq 2$;
- b) $p \in (\max\{1, \alpha\}, 2)$ and $b(t, x) \equiv 0$.

$$\left\| \sup_{t_i \in \Pi} |X_{t_i} - X_{t_i}^\tau| \right\|_{L^p(\Omega)} \lesssim \mathcal{R}_p(n, h) + h^\theta, \quad \theta = \begin{cases} \frac{\epsilon(p-\alpha)}{\alpha p}, & p < 2 \\ \frac{\epsilon(2-\alpha)}{2\alpha}, & p \geq 2. \end{cases} \quad (5)$$

$\mathcal{R}_p(n, h)$: from Euler-Peano scheme, h^θ : from DC.

- Take $h = n^{-\beta/\theta}$ + restr. on $\theta \implies$

$$\left\| \sup_{t_i \in \Pi} |X_{t_i} - X_{t_i}^\tau| \right\|_{L^p(\Omega)} \lesssim n^{-\beta}, \quad \beta = \begin{cases} \min(\gamma, 1/2), & p < 2 \\ \min(\gamma, 1/p), & p \geq 2 \end{cases}$$

Comments:

- Bossy, Maurer'24: AR approximation. $p \geq 2$.
- Wear errors: also possible (D. Platonov);

Simulation comparison I:

$$X_t = \int_0^t \int_{|z| \leq 1} s^{-\sigma} \cos(X_{s-}) z \tilde{N}(ds, dz), \quad X_0 = 0, \\ \nu(dz) = \alpha M 1_{\{|z| \leq 1\}} |z|^{-1-\alpha} dz, \quad N(r) = Mr^{-\alpha}.$$

$$DC : \quad \mathcal{NJ} = 2\lambda_h(1) = \frac{2}{1-\epsilon} h^{-\epsilon} \implies h = \left(\frac{\mathcal{NJ}(1-\epsilon)}{2} \right)^{-1/\epsilon}, \quad (6)$$

$$AR : \quad \mathcal{NJ} := N(\delta_{AR}) = \frac{2M}{\delta_{AR}^\alpha} \implies \delta_{AR} = \left(\frac{2M}{\mathcal{NJ}} \right)^{1/\alpha}. \quad (7)$$

$$R_t^\tau(\epsilon) = \int_0^t \int_{B(s,h)} s^{-\sigma} z \tilde{N}(ds, dz).$$

Now fixing $\mathcal{NJ}$ we find the ϵ , which minimizes the variance of R_t^τ :

$$\text{Var } R_t^\tau(\epsilon) = 2 \int_0^1 \int_0^{\tau((sh)^\epsilon)} s^{-2\sigma} z^2 \nu(dz) ds \mapsto \min \\ \implies \arg \min_\epsilon \text{Var } R_t^\tau(\epsilon) = \alpha\sigma.$$

Fine level $\mathcal{NJ}^{\text{fine}} = 64,000$ jumps, and a coarse level $\mathcal{NJ}^{\text{coarse}} = 4,000$ jumps.

1 **Fine parameters:** $\mathcal{NJ} = \mathcal{NJ}^{\text{fine}}, (\delta_{AR}^{\text{fine}}, h^{\text{fine}}).$

2 **Coarse parameters:** $\mathcal{NJ} = \mathcal{NJ}^{\text{coarse}}, (\delta_{AR}^{\text{coarse}}, h^{\text{coarse}}).$

Simulation comparison II

- a) generate a single *fine* path using the fine parameters;
- b) derive the *coarse* path by filtering the jumps from this fine path using the coarse parameters.

Procedure:

- 1 Generate a single *fine* path on $[0, 1]$ with an expected $\mathcal{N}\mathcal{J}^{\text{fine}}$ jumps, using the calculated $(\delta_{\text{AR}}^{\text{fine}}, h^{\text{fine}})$ parameters.
- 2 From this fine path, build a *coarse* path by retaining a jump at time s only if it exceeds the coarse-level threshold:
 - **DC method:** The jump is kept if $|z| \geq \tau((sh^{\text{coarse}})^\epsilon)$.
 - **AR method:** The jump is kept if $|z| \geq \delta_{\text{AR}}^{\text{coarse}}$.

Let $Y := \sup_{0 \leq t \leq 1} |X_t^{\text{fine}} - X_t^{\text{coarse}}|$, and define i.i.d. copies

$$Y_k = \sup_{0 \leq t \leq 1} |X_t^{\text{fine},(k)} - X_t^{\text{coarse},(k)}|, \quad 1 \leq k \leq D.$$

Estimate $\bar{Y}$ and construct confidence intervals.

Comparison of simulation results for AR and DC ($\sigma = 0.25$)

α	AR Method			DC Method		
	Error	95% CI		Error	95% CI	
		Lower	Upper		Lower	Upper
<i>Small α</i>						
0.10	4.952e−36	3.789e−36	6.270e−36	2.657e-36	7.878e−37	5.628e−36
0.20	2.025e−17	1.988e−17	2.063e−17	1.614e-17	1.581e−17	1.648e−17
0.30	2.750e−11	2.726e−11	2.773e−11	2.470e-11	2.449e−11	2.491e−11
0.40	3.317e−08	3.289e−08	3.346e−08	2.968e-08	2.942e−08	2.994e−08
<i>Middle α</i>						
0.50	2.434e−06	2.413e−06	2.455e−06	2.172e-06	2.153e−06	2.192e−06
1.00	1.582e−02	1.566e−02	1.598e−02	1.413e-02	1.398e−02	1.428e−02
1.50	2.884e−01	2.850e−01	2.919e−01	2.644e-01	2.613e−01	2.676e−01
<i>Large α</i>						
1.60	3.938e−01	3.889e−01	3.986e−01	3.665e-01	3.620e−01	3.709e−01
1.70	4.978e−01	4.919e−01	5.037e−01	4.698e-01	4.642e−01	4.755e−01
1.80	5.766e−01	5.700e−01	5.832e−01	5.529e-01	5.466e−01	5.592e−01
1.90	5.751e−01	5.693e−01	5.810e−01	5.618e-01	5.560e−01	5.676e−01

Small jumps vs Normal

Distance between small-jump Lévy-type process and Gaussian.

- Uniform grid on $[0, t]$.

Suppose we generated X_t by the Euler scheme. Fix $\vartheta := (X_{t_0}^\tau, \dots, X_{t_n}^\tau)$.

Consider the “modified” small-jump part

$$\mathcal{J}_h(t) := \int_0^{\eta(t)} \int_{B(s,h)} c(s, X_{\eta(s-)}^\tau, z) \tilde{N}(ds, dz). \quad (8)$$

Proposition

Assume $\epsilon + 2\sigma < 1$. $\forall t \in (0, T]$ there exists an $\mathcal{F}_t$ – measurable r.v. $\zeta_h(t)$, $\text{Law}(\zeta_h(t)|\vartheta) = \mathcal{N}(0, \sigma_{h,\theta}^2(t))$, where $\sigma_{h,\theta}^2(t) = \text{Var}(\mathcal{J}_h(t)|\theta)$ is the conditional variance, s.t. for any $p \geq 1$ and some $\chi \in (0, 1)$,

$$\|\mathcal{J}_h(t) - \zeta_h(t)\|_{L^p(\Omega)} \lesssim \tau(h^\epsilon) h^{-\frac{\epsilon}{2}} \left(h^{\frac{\epsilon}{2}} + n^{-\frac{1}{2}} \right)^{\frac{\chi}{p}}. \quad (9)$$

What for:

$$p \geq 2, \tau(t) \approx t^{1/\alpha} \implies$$

$$\left\| \sup_{t_i \in \Pi} |X_{t_i} - X_{t_i}^\tau| \right\|_{L^p(\Omega)} \lesssim \mathcal{R}_p(n, h) + h^{\frac{\epsilon(2-\alpha)}{2\alpha}},$$

$$\|\mathcal{J}_h(t) - \zeta_h(t)\|_{L^p(\Omega)} \lesssim h^{\frac{\epsilon(2-\alpha)}{2\alpha}} \left(h^{\frac{\epsilon}{2}} + n^{-\frac{1}{2}} \right)^{\frac{\chi}{p}}.$$

- “Gaussian error” is dominated by the first order appr. error.
- adding ζ_h as a corrector does not improve the order of convergence
- restoring small jumps: improves the distributional properties of
- $\mathcal{J}_h$ vs. ζ_h : simulating ζ_h is more time-efficient.
- Hint for the parametrix: why not to use Gaussian proxi

Thank you!