

Numerical simulation of fractional-noise SDEs for all Hurst parameters via Wick–Itô–Skorohod integration

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1. Introducing the SDE and convergence result
2. How to interpret $\int Y(t) dB^H(t)$
3. Basic tools in Wick calculus
4. Derivation of the scheme
5. Numerical illustrations
6. Conclusion

Examine SDE with multiplicative noise

$$dX(t) = \alpha X(t)dt + a(t, X(t))dt + \beta X(t)dB^H(t).$$

Here B^H is fractional Brownian motion (fBm).

Need to understand

$$\int X(t) dB^H(t)$$

for some stochastic process X .

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Young Integral: $H > 1/2$.

Rough paths: $H > 1/3$, $H > 1/4$.

Skorohod: $H \in (0, 1)$

► Wick–Itô–Skorohod (WIS) integral valid for all $H \in (0, 1)$.

Conclusion:

Consider SDE:

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► Numerical evidence suggests the sharper rate

$$\left(\mathbb{E} \left[(X(t_n) - X_n)^2 \right]\right)^{1/2} \leq C_H \Delta t^{\min(H + \frac{1}{2}, 1)}$$

for $H \in (0, 1)$.

Recall Fractional Brownian motion (fBm)

Take Hurst parameter $H \in (0, 1]$.

► *Fractional Brownian motion* $(B^H(t))_{t \geq 0}$: centered and continuous Gaussian process with the covariance function

$$\mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2} \left(t^{2H} + s^{2H} - |t - s|^{2H} \right).$$

See $\mathbb{E}[B^H(0)^2] = 0$, so $B^H(0) = 0$ a.s. and $\mathbb{E}[B^H(t)^2] = t^{2H}$.

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When $H = \frac{1}{2}$, we have for $t, s \geq 0$ that

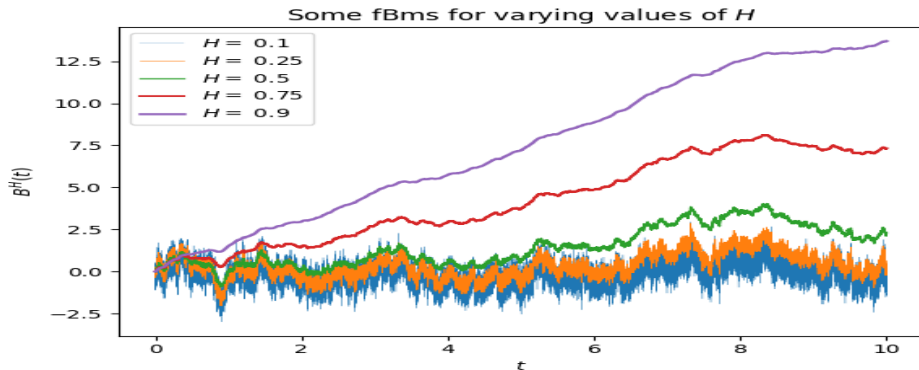
$$\mathbb{E}[B^{1/2}(t)B^{1/2}(s)] = \frac{1}{2}(t + s - |t - s|) = t \wedge s.$$

Therefore, fBm is standard Brownian motion for $H = \frac{1}{2}$.

Increments of fBm are stationary. However, they need not be independent.

- ▶ For $H > \frac{1}{2}$, the increments of an fBm are positively correlated.
- ▶ For $H < \frac{1}{2}$, the increments of an fBm are negatively correlated.
- ▶ For $H = \frac{1}{2}$, the increments of an fBm are uncorrelated.

fBm can be sampled efficiently using the circulant embedding method.



WIS Integral

We use the white-noise/Wick construction, the WIS integral, for $H \in (0, 1)$, where $W^H = dB^H/dt$ in the sense of stochastic distributions:

$$\int Y(t)dB^H(t) := \int Y(t) \diamond W^H(t)dt.$$

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Let $\{\xi_k\}$ be the Hermite basis of $L^2(\mathbb{R})$ and $\varepsilon_k = (0, \dots, 1, \dots)$. Here $\mathcal{H}_{\varepsilon_k}(\omega) = \langle \omega, \xi_k \rangle$ is the k th first-chaos variable. Then

$$B^H(t) = \sum_{k=1}^{\infty} \left[\int_0^t (M_H \xi_k)(s) ds \right] \mathcal{H}_{\varepsilon_k}.$$

The operator M_H can be defined in several ways but is designed such that

$$\langle M_H \mathbf{1}_{[0,t]}, M_H \mathbf{1}_{[0,s]} \rangle_{L^2(\mathbb{R})} = \frac{1}{2} \left(|t|^{2H} + |s|^{2H} - |t-s|^{2H} \right).$$

$$W^H(t) := \sum_{k=1}^{\infty} (M_H \xi_k)(t) \mathcal{H}_{\varepsilon_k}.$$

This is a Wiener–Itô chaos expansion; $\diamond$ denotes the Wick product.

Wick Product

Let $\mathcal{J}$ be the set of finite multi-indices and h_n the n th Hermite polynomial. For $\alpha \in \mathcal{J}$, define $\mathcal{H}_\alpha(\omega) := \prod_{j \geq 1} h_{\alpha_j}(\langle \omega, \xi_j \rangle)$. Then $\{\mathcal{H}_\alpha\}_{\alpha \in \mathcal{J}}$ is an orthogonal basis of $L^2(\mathbb{P})$.

Definition

Let $F = \sum_{\alpha \in \mathcal{J}} a_\alpha \mathcal{H}_\alpha$ and $G = \sum_{\beta \in \mathcal{J}} b_\beta \mathcal{H}_\beta$. The *Wick product* is the bivariate mapping

$$F \diamond G = \sum_{\alpha, \beta \in \mathcal{J}} a_\alpha b_\beta \mathcal{H}_{\alpha+\beta}.$$

Powers with respect to the Wick product, denoted by $F^{\diamond n}$, are defined by induction:

$$F^{\diamond n} := \begin{cases} F \diamond F^{\diamond(n-1)}, & \text{if } n \geq 2, \\ F, & \text{if } n = 1. \end{cases}$$

We set $F^{\diamond 0} = 1$ for convenience.

Wick Calculus

Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is an analytic function with Taylor series $f(x) = \sum_{n=0}^{\infty} a_n x^n$.
The *Wick version* of f evaluated at X is

$$f^{\diamond}(X) = \sum_{n=0}^{\infty} a_n X^{\diamond n}.$$

Wick chain rule

Let Z be continuously differentiable in $(S)^*$ and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be analytic. Then

$$\frac{d}{dt} f^{\diamond}(Z(t)) = (f')^{\diamond}(Z(t)) \diamond \frac{d}{dt} Z(t).$$

Wick product rule

Let X and Y both be differentiable in $(S)^*$. Then $X(t) \diamond Y(t)$ is also differentiable, with

$$\frac{d}{dt} (X(t) \diamond Y(t)) = \left(\frac{d}{dt} X(t) \right) \diamond Y(t) + X(t) \diamond \left(\frac{d}{dt} Y(t) \right).$$

The Wick exponential

$$\exp^{\diamond}(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^{\diamond n}.$$

The Wick exponential shares some properties with the regular exponential:

$$\exp^{\diamond}(F + G) = \exp^{\diamond}(F) \diamond \exp^{\diamond}(G).$$

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- Generally, the Wiener–Itô chaos expansion of a random variable is not known, so calculating its Wick powers is tricky.
- Even if the expansion is known, the infinite sum might make it very complicated to evaluate the Wick exponential.
- For $w_f(\omega) = \langle \omega, f \rangle$, $f \in L^2(\mathbb{R})$, its Wick exponential can be expressed as a regular exponential:

$$\exp^\diamond(w_f) = \exp\left(w_f - \frac{1}{2}\|f\|_{L^2(\mathbb{R})}^2\right).$$

$$\int Y(t)dB^H(t) := \int Y(t) \diamond W^H(t)dt.$$

The integral is centered, i.e.

$$\mathbb{E} \left[\int Y dB^H \right] = 0.$$

The WIS integral extends the Itô integral when $H = \frac{1}{2}$, and the two coincide.
For example,

$$\int_0^t B^H(s)dB^H(s) = \frac{1}{2}B^H(t)^2 - \frac{1}{2}t^{2H}.$$

Translation operator:

$$(T_{\omega_0}X)(\omega) := X(\omega + \omega_0), \quad \text{for all } \omega \in \Omega.$$

Gjessing's Lemma

Let $p > 1$. Suppose that $f \in L^2(\mathbb{R})$ and $X \in L^p(\mathbb{P})$.

Then $X \diamond \exp^\diamond(w_f) \in L^q(\mathbb{P})$ for all $q < p$ and

$$X \diamond \exp^\diamond(w_f) = (T_{-f}X) \cdot \exp^\diamond(w_f)$$

almost surely. So

$$X \diamond \exp^\diamond(w_f) = (T_{-f}X) \cdot \exp\left(w_f - \frac{1}{2}\|f\|_{L^2(\mathbb{R})}^2\right).$$

Semilinear SDEs

Consider, given $X(0) = x_0$ and $t \in [0, T]$,

$$dX(t) = \alpha X(t)dt + a(t, X(t))dt + \beta X(t)dB^H(t),$$

where $\alpha, \beta, x_0 \in \mathbb{R}$, $T > 0$ and $a : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$.

Interpret this equation as the differential equation

$$\frac{d}{dt}X(t) = \alpha X(t) + a(t, X(t)) + \beta X(t) \diamond W^H(t), \quad X(0) = x_0, \quad t \in [0, T].$$

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To ensure existence and uniqueness of the solution, assume that $a : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies two conditions:

1. *Linear growth*: there exists $C > 0$ such that $|a(t, x)| \leq C(1 + |x|)$ for all $t \in [0, T]$, $x \in \mathbb{R}$.
2. *Lipschitz continuity*: there exists $C > 0$ such that $|a(t, x) - a(t, y)| \leq C|x - y|$ for all $t \in [0, T]$, $x, y \in \mathbb{R}$.

Assume $X(t)$ is a solution of the semilinear SDE

$$\frac{d}{dt}X(t) = \alpha X(t) + a(t, X(t)) + \beta X(t) \diamond W^H(t)$$

. Consider the factor

$$J(t) := \exp^\diamond(-\alpha t - \beta B^H(t)).$$

By the Wick chain rule,

$$\frac{d}{dt}J(t) = J(t) \diamond (-\alpha - \beta W^H(t)).$$

Define $Z(t) := J(t) \diamond X(t)$. From Z we can recover X .

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By the Wick product rule and since $X(t)$ solves the SDE,

$$\frac{d}{dt}Z(t) = \left(\frac{d}{dt}J(t)\right) \diamond X(t) + J(t) \diamond \left(\frac{d}{dt}X(t)\right) = J(t) \diamond a(t, X(t)).$$

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Gjessing's Lemma gives

$$\frac{d}{dt}Z(t) = J(t) \left(T_{\beta M_H \mathbf{1}_{[0,t]}} a(t, X(t)) \right).$$

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Recall $Z(t) := J(t) \diamond X(t)$. Gjessing's Lemma again gives

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Inverting for X , we have

$$X(t) = T_{-\beta M_H \mathbf{1}_{[0,t]}} \left[J(t)^{-1} Z(t) \right].$$

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Using this with the first equation, we get

$$\frac{d}{dt}Z(t) = J(t) a(t, J(t)^{-1} Z(t)).$$

This gives a unique solution Z , and hence a unique solution X .

It is also the basis for the numerical method.

Recovering $X(t)$ from $Z(t) = J(t) \diamond X(t)$: a target-dependent ODE

Fix a target time t . For $0 \leq s \leq t$, define

$$f_t = -\beta M_H \mathbf{1}_{[0,t]}, \quad \hat{J}^t(s) = T_{f_t} J(s), \quad \hat{Z}^t(s) = T_{f_t} Z(s).$$

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Recover the target value

Translation respects products and inverses, so

$$\begin{aligned} X(t) &= T_{f_t} [J(t)^{-1} Z(t)] \\ &= (T_{f_t} J(t))^{-1} T_{f_t} Z(t) \\ &= \boxed{\frac{\hat{Z}^t(t)}{\hat{J}^t(t)}}. \end{aligned}$$

Translate the random ODE

Applying T_{f_t} gives

$$\boxed{\begin{aligned} \frac{d}{ds} \hat{Z}^t(s) &= \hat{J}^t(s) a\left(s, \frac{\hat{Z}^t(s)}{\hat{J}^t(s)}\right), \\ \hat{Z}^t(0) &= x_0. \end{aligned}}$$

t : fixed target time

s : running ODE time

The translated integrating factor is explicit

Translation of the fBm path

Using

$$B^H(s, \omega) = \langle \omega, M_H \mathbf{1}_{[0,s]} \rangle,$$

we obtain

$$T_{f_t} B^H(s) = B^H(s) - \beta R_H(t, s),$$

where

$$R_H(t, s) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).$$

Translated integrating factor

Since

$$J(s) = \exp\left(-\alpha s - \frac{\beta^2}{2} s^{2H} - \beta B^H(s)\right),$$

for $0 \leq s \leq t$,

$$\widehat{J}^t(s) = \exp\left[-\alpha s + \frac{\beta^2}{2} (t^{2H} - (t-s)^{2H}) - \beta B^H(s)\right].$$

The noise translation becomes a deterministic covariance correction.

GfBm Euler–Maruyama algorithm

Let $t_i = i\Delta t$, $\Delta t = T/n$, $\tilde{J}_i^k = \hat{J}^{t_k}(t_i)$, and $\tilde{Z}_i^k \approx \hat{Z}^{t_k}(t_i)$.

Input and output

Input: H, x_0, T, n and $B_i^H = B^H(t_i)$, $i = 0, \dots, n$.

Output: $X_k \approx X(t_k)$, $k = 1, \dots, n$.

For each target index $k = 1, \dots, n$:

1. Initialize $\tilde{Z}_0^k = x_0$.
2. For $i = 0, \dots, k-1$, evaluate and update

$$\tilde{J}_i^k = \exp \left[-\alpha i \Delta t + \frac{\beta^2}{2} \Delta t^{2H} (k^{2H} - (k-i)^{2H}) - \beta B_i^H \right],$$

$$\tilde{Z}_{i+1}^k = \tilde{Z}_i^k + \Delta t \tilde{J}_i^k a \left(i \Delta t, \frac{\tilde{Z}_i^k}{\tilde{J}_i^k} \right).$$

3. At the target time, evaluate

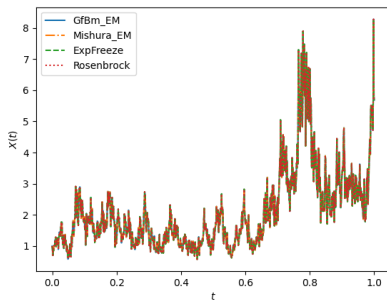
$$\tilde{J}_k^k = \exp \left[-\alpha k \Delta t + \frac{\beta^2}{2} (k \Delta t)^{2H} - \beta B_k^H \right],$$

$$X_k = \frac{\tilde{Z}_k^k}{\tilde{J}_k^k}.$$

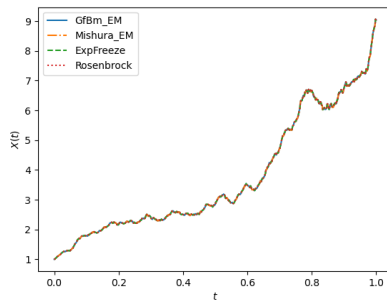
Mishura EM versus GfBm EM, Exponential Freeze & Rosenbrock

$$dX = \left(X + \frac{4X}{1 + X^2} \right) dt + X dB^H, \quad X(0) = 1.$$

$H = 0.25$



$H = 0.75$



We prove for our method, for all $H \in (0, 1)$,

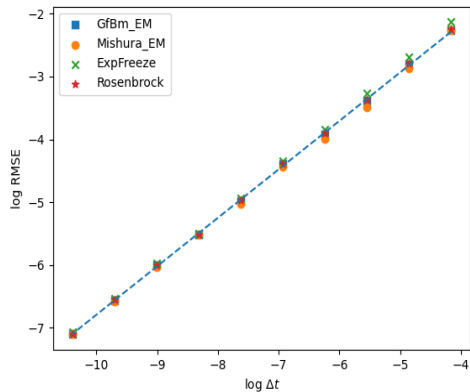
$$\left(\mathbb{E} \left[(X_n - X(t_n))^2 \right] \right)^{1/2} \leq C_H \Delta t^H$$

for some $C_H > 0$.

This extends Mishura's result (Mishura, 2008) to $H \in (0, 1)$.

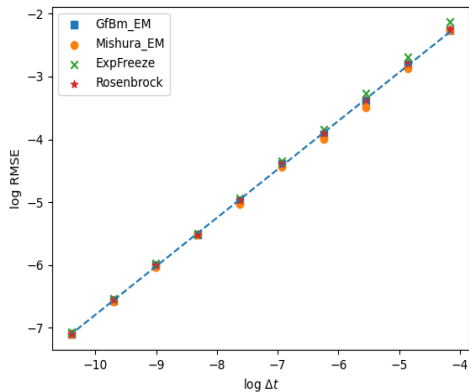
To our knowledge, the required target-dependent translation had not previously been implemented.

Convergence



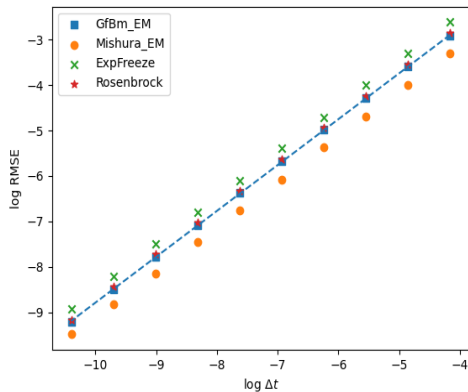
$H = 0.25$: fitted slope $0.771 \approx H + \frac{1}{2}$.

Convergence



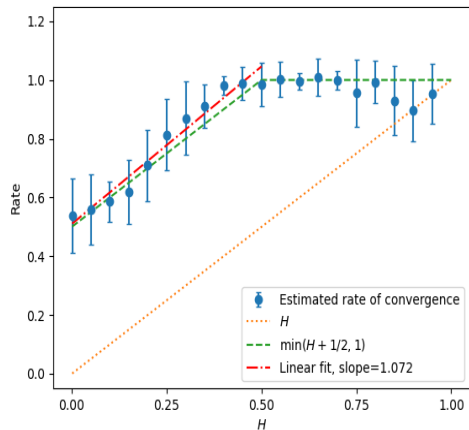
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Rates look better than the theoretical rate H .



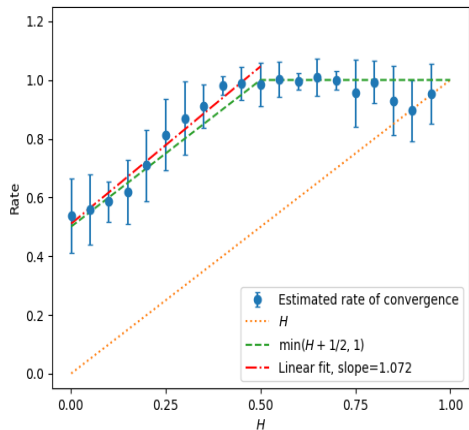
$H = 0.75$: fitted slope 1.009.

Rates of convergence vs H

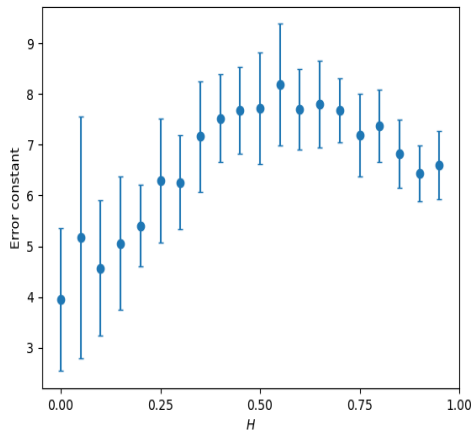


Estimated strong rate.

Rates of convergence vs H



Estimated strong rate.



Estimated error constant.

For Euler-type numerical methods we prove for $H \in (0, 1)$

$$\left(\mathbb{E} \left[(X(t_n) - X_n)^2 \right] \right)^{1/2} \leq C_H \Delta t^H.$$

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$$\left(\mathbb{E} \left[(X(t_n) - X_n)^2 \right] \right)^{1/2} \leq C_H \Delta t^H.$$

► We conjecture it should be

$$\left(\mathbb{E} \left[(X(t_n) - X_n)^2 \right] \right)^{1/2} \leq C_H \Delta t^{\min(H + \frac{1}{2}, 1)}$$

for $H \in (0, 1)$.

(Examined a number of globally Lipschitz functions and large noise.)

arXiv:2404.07013

- Proving sharper rate.

Needs a different approach: Malliavin calculus ...?

- Non-globally Lipschitz drift and nonlinear diffusion.

Existence-Uniqueness ; Tamed - Truncated Methods.

- Systems: $A \in \mathbb{R}^{d \times d}$ or SPDE $A = \Delta$:

$$dU(t) = (AU(t) + F(U)) dt + U(t) dB^H(t).$$

1. WIS integration provides a framework for all $H \in (0, 1)$.
2. Explicit noise translation makes the target-dependent random ODE implementable.
3. Strong order H is proved; numerics suggest $\min\left(H + \frac{1}{2}, 1\right)$.

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