

Global Solutions in Stochastic Fluid Dynamics Models

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 **STOCHASTICA**

Overview

- ▶ Motivation & Heuristics: preventing blow-up of solutions to (S)PDEs
- ▶ Assumptions
- ▶ Uniform control
- ▶ Steps of the main proof
- ▶ Final remarks

Joint work with Dan Crisan (Imperial College London), based on the paper:

O. Lang, D. Crisan, Global solutions for stochastically controlled fluid dynamics models. Stoch PDE: Anal Comp (2025).

Motivation: preventing blow-up

- ▶ For a large class of models from geophysical fluid dynamics, only local in time existence of strong solutions holds (deterministic & stochastic) e.g. 3D Euler equation, rotating shallow water, etc.
 - ▶ For transport models, this is sometimes due to the intrinsic structure of the nonlinear transport term (and the transport noise term e.g. $\xi \cdot \nabla \omega$).
- ▶ We look for a stochastic term which can prevent the blow-up, when the drift is nonlinear - typically of 'Navier-Stokes' type e.g. $u \cdot \nabla u$, $u \cdot \nabla \omega$.
- ▶ The associated SPDE:

$$dX_t = A_t(X_t)dt + B_t(X_t)dW_t \quad (1)$$

where B is a nonlinear operator.

Informal framework and strategy

- ▶ In fluid models: typically a delicate balance between the energy created within the multi-scale system and the energy dissipated through the diffusion term - when this is broken → **blow-up**.
- ▶ Broader approach: we aimed not just to suppress the blow-up, but to ensure existence of a solution when no solution is known to exist in the original deterministic case.

⇒ stochastic closure for e.g. a compressible inviscid model (no div-free condition and no diffusion to close the system as in e.g. rotating shallow water equations).

Covered: compressible and incompressible, viscous and inviscid models.

- ▶ Fine balance between: conditions imposed on the nonlinear drift, initial conditions, and the choice of the stochastic perturbation
→ 3 distinct cases dictated by the form of the nonlinear drift and the required regularity of the solution.

Literature review

- ▶ It all started with:
 - a. *Well-posedness for a stochastic 2D Euler equation with transport noise*, *Stochastic PDE: Analysis and Computation* 11, 433–480 (2023) (w. D. Crisan)
 - b. *Well-posedness properties for a stochastic rotating shallow water model*. *J. Dyn. Diff. Equat.*, 36, 3175–3205 (2024) (w. D. Crisan)
 - c. *Analytical properties for a stochastic rotating shallow water model under location uncertainty*, *Journal of Mathematical Fluid Mechanics*, 25, Article 29 (2023) (w. D. Crisan, E. Mémin)
- ▶ Our result is closest in spirit to *Wu & Hu, Suppression and stabilisation of noise*, *Int. J. Control* 82(11), 2150–2157 (2009)
 - existence of global solutions in a finite-dimensional setting
 - we extend this to the infinite-dimensional case, and adapt it to a general abstract functional setting.
- ▶ For the particular case of the Euler equation → *Bagnara, Maurelli, Xu, No blow-up by nonlinear Itô noise for the Euler equations*, *Electron. J. Probab.* 30, 1–29 (2025).

Heuristics

- ▶ Consider the following linear one-dimensional ODE:

$$df_t = af_t dt, \quad f_0, \quad a > 0.$$

⇒ the solution of this ODE does not blow-up, but it does increase exponentially fast in time: $f_t = f_0 \exp(at)$

- ▶ Let's compare with the solution of the same equation but now stochastically perturbed:

$$df_t = af_t dt + bf_t dW_t, \quad f_0, \quad a, b > 0,$$

⇒ we obtain a one dimensional geometric Brownian motion with the explicit solution

$$f_t = f_0 \exp \left(\left(a - \frac{b^2}{2} \right) t + bW_t \right).$$

- ▶ As soon as the diffusion coefficient is sufficiently high (we inject sufficient noise into the system), more precisely if we choose $b^2 > 2a$
⇒ the solution converges to 0 asymptotically.

Assumptions

- ▶ Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ be a fixed stochastic basis. Let $\mathcal{D}, \mathcal{F}_1, \mathcal{F}_0, \mathcal{G}, \mathcal{U}$ be separable Hilbert spaces such that

$$\mathcal{D} \hookrightarrow \mathcal{F}_1 \hookrightarrow \mathcal{F}_0 \hookrightarrow \mathcal{G} \tag{2}$$

are compact embeddings.

- ▶ In general, the main *space of existence* is $\mathcal{F}_0$, endowed with the uniform norm in time, when a higher order integral of the $\mathcal{F}_1$ norm also exists.
- ▶ The space $\mathcal{D}$ is a higher order space in which we can control the norm of an approximating sequence of solutions uniformly.
- ▶ $\mathcal{G}$ is the space in which the approximating sequence of solutions converges to the solution of the original equation.

Formal framework

- Consider the following equation on $\mathcal{G}$:

$$dX_t = A_t(X_t)dt + \theta \|X_t\|_{\mathcal{F}_i}^{\alpha_i} X_t dW_t \quad (3)$$

where

$$A : [0, T] \times \mathcal{F}_i \times \Omega \rightarrow \mathcal{G}$$

$$B : [0, T] \times \mathcal{F}_i \times \Omega \rightarrow L_2(\mathcal{U}, \mathcal{F}_i)$$

are progressively measurable with respect to $\mathcal{B}([0, t]) \times \mathcal{B}(\mathcal{F}_i) \times \mathcal{F}_t$ for any $t \in [0, T]$ and $T \in [0, \infty)$ fixed, and W is a $\mathcal{U}$ -valued Wiener process, X_0 is a $\mathcal{F}_0$ -measurable $\mathcal{F}_i$ -valued random variable, $\alpha_i, \theta \in \mathbb{R}_+$.

- We assume that this equation admits a unique maximal strong solution $(X, \mathcal{T})$ in $\mathcal{F}_i$. We have either $i = 0$ or $i = 1$, depending on the structure of the drift A .
- A global solution for equation (3) is constructed using a d -dimensional projected equation:

$$dX_t^d = A_t^d(X_t^d)dt + \theta \|X_t^d\|_{\mathcal{F}_i}^{\alpha_i} X_t^d dW_t^d \quad \text{on } [0, T] \quad (4)$$

where X_t^d is the classical d -dimensional (Galerkin) projection of X_t .

The Three Cases

- **Case I:** The initial condition $X_0 \in \mathcal{F}_0$, $\theta > 0$, assumption (A1) is satisfied, and

$$B_t(X_t) = \theta \|X_t\|_{\mathcal{F}_0}^{\alpha_0} X_t.$$

- **Case II:** The initial condition $X_0 \in \mathcal{D}$, $\theta > 0$, assumption (A2) is satisfied, and

$$B_t(X_t) = \theta \|X_t\|_{\mathcal{F}_1}^{\alpha_1} X_t.$$

- **Case III:** The initial condition $X_0 \in \mathcal{F}_1$, $\theta > 0$, assumption (A3) is satisfied, and

$$B_t(X_t) = \theta \|X_t\|_{\mathcal{F}_0}^{\alpha_0} X_t.$$

$$\mathcal{D} \hookrightarrow \mathcal{F}_1 \hookrightarrow \mathcal{F}_0 \hookrightarrow \mathcal{G}$$

Remark: The role of the space $\mathcal{F}_1$ in Case III is, in a certain sense, different from the role of the same space in the first two cases:

In Case III $\mathcal{F}_1$ **is** the main space of existence of the solution, and the noise depends on the norm of the solution in a larger space, that is the $\mathcal{F}_0$ norm. We could choose the space of existence to be $\mathcal{F}_0$ as in the previous two cases, but then we would need to introduce a new larger space between $\mathcal{F}_0$ and $\mathcal{G}$, say $\mathcal{F}^*$ which would be used to construct the stochastic term.

- ▶ The stochastic term is chosen to act on the norm of the solution, via its quadratic variation ($\rightarrow$ **second order effect**), to prevent the blow-up.

Assumptions

Assumption (A1): The initial condition is in $\mathcal{F}_0$ and for $a \in \mathcal{D}^d$ it holds that

$$\langle a, A^d(a) \rangle_{\mathcal{F}_0} \leq C_1 \|a\|_{\mathcal{F}_0}^{\gamma_1} - C_2 \|a\|_{\mathcal{F}_1}^2 \quad (5)$$

where C_1, γ_1 are positive constants and $C_2 \in \mathbb{R}$.

For $a \in \mathcal{F}_1$

$$\|a\|_{\mathcal{F}_0} \leq C_0 \|a\|_{\mathcal{F}_1}^{\alpha} \|a\|_{\mathcal{G}}^{\beta} \quad (6)$$

and

$$\|A^d(a)\|_{\mathcal{G}} \leq C_1 \|a\|_{\mathcal{F}_0}^{\gamma_1} + C_2 \|a\|_{\mathcal{F}_1}^{\gamma_2} + C_3, \quad (7)$$

where $\alpha, \beta, \gamma^i, C_i, i \in \{0, 1, 2, 3\}$ are positive constants. For any $a, b \in \mathcal{F}_0$

$$\|A^d(a) - A^d(b)\|_{\mathcal{G}} \leq C(1 + \|a\|_{\mathcal{F}_1} + \|b\|_{\mathcal{F}_1})\|a - b\|_{\mathcal{F}_0}. \quad (8)$$

with C a positive constant.

Case I: Diffusion coefficient given by

$$B_t(X_t) = \theta \|X_t\|_{\mathcal{F}_0}^{\alpha_0} X_t \quad (9)$$

Application: viscous compressible and incompressible models e.g. incompressible vorticity equation on the 3D torus, viscous 2D rotating shallow water model.

Assumptions

Assumption (A2): The initial condition is in $\mathcal{D}$, (5), (8) are true, and the following conditions hold:

$$\langle a, A^d(a) \rangle_{\mathcal{D}} \leq C_1 \|a\|_{\mathcal{F}_1}^{\gamma_1} \|a\|_{\mathcal{D}}^{\gamma_2} \quad (10)$$

$$\|a\|_{\mathcal{F}_1} \leq C \|a\|_{\mathcal{D}}^{\alpha} \|a\|_{\mathcal{G}}^{\beta} \quad (11)$$

where $C_1, C, \gamma_1, \gamma_2, \alpha, \beta$ are positive constants. Condition (7) holds with $C_2 = 0$. We also assume that the approximate equation (4) admits a solution in $\mathcal{F}_0$ which does not hit zero or, if it hits zero, it remains zero.

Case II: The diffusion coefficient is given by

$$B_t(X_t) = \theta \|X_t\|_{\mathcal{F}_1}^{\alpha_1} X_t \quad (12)$$

Application: inviscid compressible models e.g. Burgers' equation on the 1D/2D torus, 2D inviscid rotating shallow water model.

Assumptions

Assumption (A3): The initial condition is in $\mathcal{F}_1$ and the following conditions hold:

$$\langle a, A^d(a) \rangle_{\mathcal{F}_1} \leq C_1 \|a\|_{\mathcal{F}_0}^{\gamma_{13}} \|a\|_{\mathcal{F}_1}^2 \quad (13)$$

$$\|a\|_{\mathcal{F}_0} \leq C \|a\|_{\mathcal{F}_1}^{\alpha} \|a\|_{\mathcal{G}}^{\beta} \quad (14)$$

where $C_1, C, \gamma_{13}, \alpha, \beta$ are positive constants. Condition (7) holds with $C_1 = 0$. Moreover, for any $a, b \in \mathcal{F}_0$

$$\|A^d(a) - A^d(b)\|_{\mathcal{G}} \leq C(1 + \|a\|_{\mathcal{F}_1} + \|b\|_{\mathcal{F}_1}) \|a - b\|_{\mathcal{F}_1}. \quad (15)$$

with C a positive constant.

Case III: The diffusion coefficient is given by

$$B_t(X_t) = \theta \|X_t\|_{\mathcal{F}_0}^{\alpha_0} X_t \quad (16)$$

Application: inviscid incompressible models e.g. incompressible Euler equation in vorticity form on the 3D torus.

Main Results

Theorem

Under assumptions (A1) – (A3), the stochastic partial differential equation (3)

$$dX_t = A_t(X_t)dt + \theta \|X_t\|_{\mathcal{F}_t}^{\alpha_i} X_t dW_t$$

admits a global strong solution X with paths that belong to the space $C([0, \infty), \mathcal{G})$, $\mathbb{P}$ -almost surely, such that for any $T > 0$,

$$\mathbb{P} \left(\sup_{t \in [0, T]} \|X_s\|_{\mathcal{G}}^2 < \infty \right) = 1.$$

Main Results

In each of the three cases, there are stronger bounds for the process X , that hold for any $T > 0$, as follows:

$$\textbf{Case I : } \mathbb{P} \left(\sup_{t \in [0, T]} \|X_s\|_{\mathcal{F}_0}^2 + \int_0^T \|X_s\|_{\mathcal{F}_1}^2 ds < \infty \right) = 1$$

$$\textbf{Case II : } \mathbb{P} \left(\sup_{t \in [0, T]} \|X_s\|_{\mathcal{D}}^2 < \infty \right) = 1$$

$$\textbf{Case III : } \mathbb{P} \left(\sup_{t \in [0, T]} \|X_s\|_{\mathcal{F}_1}^2 < \infty \right) = 1.$$

Example: 3D vorticity equation

Fluid vorticity ω , fluid velocity u , on the 3D torus $(\mathbb{T}^3)$, with $\nabla \cdot u = 0$,

$$d\omega + u \cdot \nabla \omega dt - \omega \cdot \nabla u dt = \Delta \omega dt \quad (17)$$

$$d\omega = A(\omega)dt.$$

where

$$A(\omega) := -u \cdot \nabla \omega dt + \omega \cdot \nabla u dt + \Delta \omega$$

and u is obtained via the Biot-Savart operator $u = -\text{curl} \Delta^{-1} \omega$.

For the vorticity equation (17), a possible choice for the diffusion term is

$$B(\omega) = \theta \|\omega\|_{W^{2,2}}^\alpha \omega$$

with θ, α positive and $\mathcal{G} := L^2(\mathbb{T}^3; \mathbb{R}^3)$ the space of square integrable zero-mean and divergence-free three dimensional vector fields,
 $\mathcal{F}_0 := W^{2,2}(\mathbb{T}^3; \mathbb{R}^3)$, $\mathcal{F}_1 := W^{3,2}(\mathbb{T}^3; \mathbb{R}^3)$, $\mathcal{D} := W^{4,2}(\mathbb{T}^3; \mathbb{R}^3)$.

See Case 1 above.

Example: Burger's equation

Burgers' equation on the 2D torus $(\mathbb{T}^2)$

$$d_t u + u \cdot \nabla u = 0$$

with

$$A(u) := -u \cdot \nabla u$$

and $\mathcal{G} := L^2(\mathbb{T}^3; \mathbb{R}^3)$, $\mathcal{F}_0 := W^{1,2}(\mathbb{T}^2; \mathbb{R}^2)$, $\mathcal{F}_1 := W^{3,2}(\mathbb{T}^2; \mathbb{R}^2)$ $\mathcal{D} := W^{4,2}(\mathbb{T}^2; \mathbb{R}^2)$. We have

$$B(u) = \theta \|u\|_{W^{3,2}}^\alpha u.$$

See Case 2 above.

Auxiliary results

- ▶ Revuz & Yor: If Y is a continuous local martingale which vanishes at 0, then

$$\mathbb{P}\left(\sup_t Y_t \geq x, \quad \langle Y \rangle_t \leq y\right) \leq \frac{1}{e^{-x^2/2y}}.$$

- ▶ Gawarecki & Mandrekar: Let K, H be separable Hilbert spaces, Q a trace-class operator on K and $\Phi \in L_2(K_Q, H)$. Then for arbitrary $\delta > 0$ and $n > 0$,

$$\mathbb{P}\left(\sup_{t \leq T} \left\| \int_0^t \Phi(s) dW_S \right\|_H > \delta\right) \leq \frac{n}{\delta^2} + \mathbb{P}\left(\int_0^T \|\Phi(s)\|_{L_2(K_Q, H)}^2 ds > n\right).$$

Uniform Control and Tightness

Proposition (Uniform control)

The approximating sequence $(X_t^d)_d$ is $\mathbb{P}$ -a.s. uniformly controlled in $\mathcal{F}_0$ in the uniform norm, and in $\mathcal{F}_1$ in L^2 sense. That is for any $\epsilon_1, \epsilon_2 > 0$ there exists $\mathcal{K}_1, \mathcal{K}_2 > 0$ such that

$$\sup_d \mathbb{P} \left(\sup_t \|X_t^d\|_{\mathcal{F}_0}^2 \geq \mathcal{K}_1 \right) \leq \epsilon_1 \quad (18)$$

and

$$\sup_d \mathbb{P} \left(\int_0^T \|X_s^d\|_{\mathcal{F}_1}^2 ds \geq \mathcal{K}_2 \right) \leq \epsilon_2. \quad (19)$$

Proposition - Tightness (Aldous)

The approximating sequence of solutions $(X_t^d)_d$ is tight in $C([0, T], \mathcal{G})$.

Uniform control

We first show that for any $\tilde{\epsilon} > 0$ there exists $\tilde{\mathcal{K}}$ such that

$$\sup_d \mathbb{P} \left(\sup_t \log \left(C + \|X_t^d\|_{\mathcal{F}_0}^2 \right) \geq \tilde{\mathcal{K}} \right) \leq \tilde{\epsilon}.$$

By the Itô formula we have

$$d \|X_t^d\|_{\mathcal{F}_0}^2 = \tilde{A}_t (X_t^d) dt + \tilde{B}_t (X_t^d) dW_t^d$$

where

$$\tilde{A}_t (X_t^d) := \left(2_{\mathcal{D}} \langle X_t^d, A_t^d (X_t^d) \rangle_{\mathcal{G}} + \|B_t (X_t^d)\|_{L_2(\mathcal{U}, \mathcal{F}_0)}^2 \right)$$

and

$$\tilde{B}_t (X_t^d) dW_t^d := 2 \langle X_t^d, B_t (X_t^d) dW_t \rangle_{\mathcal{F}_0}.$$

Then

$$\begin{aligned}
 d \log \left(C + \|X_t^d\|_{\mathcal{F}_0}^2 \right) &= \frac{1}{C + \|X_t^d\|_{\mathcal{F}_0}^2} \left(\tilde{A}_t^d (X_t^d) dt + \tilde{B}_t^d (X_t^d) dW_t^d \right) \\
 &\quad - \frac{1}{2 \left(C + \|X_t^d\|_{\mathcal{F}_0}^2 \right)^2} \tilde{B}_t^d (X_t^d)^2 dt \\
 &= \frac{1}{C + \|X_t^d\|_{\mathcal{F}_0}^2} \tilde{A}_t^d (X_t^d) dt - \frac{1 - \epsilon}{2 \left(C + \|X_t^d\|_{\mathcal{F}_0}^2 \right)^2} \tilde{B}_t^d (X_t^d)^2 dt \\
 &\quad + \frac{1}{C + \|X_t^d\|_{\mathcal{F}_0}^2} \tilde{B}_t^d (X_t^d) dW_t^d - \frac{\epsilon}{2 \left(C + \|X_t^d\|_{\mathcal{F}_0}^2 \right)^2} \tilde{B}_t^d (X_t^d)^2 dt.
 \end{aligned}$$

If we choose

$$M_t := \int_0^t \tilde{B}_s^d (X_s^d) dW_s$$

then

$$\langle M \rangle_t = \int_0^t \tilde{B}_s^d (X_s^d)^2 ds.$$

Now consider the three cases separately.

Case I: The diffusion term is given by

$$B_t^d(X_t^d) = \theta \|X_t^d\|_{\mathcal{F}_0}^{\alpha_0} X_t^d.$$

That is

$$\begin{aligned} \|B_t^d(X_t^d)\|_{L_2(\mathcal{U}, \mathcal{F}_0)}^2 &= \theta^2 \|X_t^d\|_{\mathcal{F}_0}^{2\alpha_0+2} \\ \tilde{B}_t^d(X_t^d)^2 &:= (2\langle X_t^d, B_t^d(X_t^d) \rangle)^2 = 4\theta^2 \|X_t^d\|_{\mathcal{F}_0}^{2\alpha_0+4} \end{aligned}$$

and

$$\begin{aligned} \tilde{A}_t^d(X_t^d) &= 2\mathcal{D}\langle X_t^d, A_t^d(X_t^d) \rangle_{\mathcal{G}} + \|B_t(X_t^d)\|_{L_2(\mathcal{U}, \mathcal{F}_0)}^2 \\ &= 2\langle X_t^d, A_t^d(X_t^d) \rangle_{\mathcal{F}_0} + \theta^2 \|X_t^d\|_{\mathcal{F}_0}^{2\alpha_0+2} \\ &\leq 2C_1 \|X_t^d\|_{\mathcal{F}_0}^{\gamma_1} - 2C_2 \|X_t^d\|_{\mathcal{F}_1}^{\gamma_2} + \theta^2 \|X_t^d\|_{\mathcal{F}_0}^{2\alpha_0+2} \end{aligned}$$

due to conditions imposed in (A1). Let us define the semimartingale

$$dM_t := 2\theta_1 \left\langle X_t^d, \|X_t^d\|_{\mathcal{F}_0}^{\alpha_1} X_t^d \right\rangle dW_t^d$$

Then

$$d \|X_t^d\|_{\mathcal{F}_0}^2 + \tilde{C}_2 \|X_t^d\|_{\mathcal{F}_1}^{\gamma_2} dt \leq G \left(\|X_t^d\|_{\mathcal{F}_0} \right) dt + dM_t$$

where $\tilde{C}_2 > 0$ and it depends on C_2 and

$$G \left(\|X_t^d\|_{\mathcal{F}_0} \right) := \tilde{C}_1 \|X_t^d\|_{\mathcal{F}_0}^{\gamma_1} + \theta^2 \|X_t^d\|_{\mathcal{F}_0}^{2\alpha_0+2}.$$

But then

$$d \|X_t^d\|_{\mathcal{F}_0}^2 \leq G \left(\|X_t^d\|_{\mathcal{F}_0} \right) dt + \tilde{B}_t (X_t^d) dW_t$$

and for $C > 0$ and

$$\tilde{C} (X_t^d) := C + \|X_t^d\|_{\mathcal{F}_0}^2$$

we can write:

$$\begin{aligned}
d \log \left(C + \|X_t^d\|_{\mathcal{F}_0}^2 \right) &\leq \frac{1}{\tilde{C} (X_t^d)^2} \left(\tilde{C} (X_t^d) G \left(\|X_t^d\|_{\mathcal{F}_0} \right) - \frac{1-\epsilon}{2} \tilde{B}_t^d (X_t^d)^2 \right) dt \\
&+ dM_t - \frac{\epsilon}{2} d\langle M \rangle_t \\
&= \frac{1}{\tilde{C} (X_t^d)^2} \left(\tilde{C} (X_t^d) \left(\tilde{C}_1 \|X_t^d\|_{\mathcal{F}_0}^{\gamma_1} + \|B_t^d (X_t^d)\|_{L_2(\mathcal{U}, \mathcal{F}_0)}^2 \right) - \frac{1-\epsilon}{2} \tilde{B}_t^d (X_t^d)^2 \right) dt \\
&+ dM_t - \frac{\epsilon}{2} d\langle M \rangle_t \\
&\dots \\
&= \frac{1}{\left(C + \|X_t^d\|_{\mathcal{F}_0}^2 \right)^2} \left(\tilde{C}_1 \|X_t^d\|_{\mathcal{F}_0}^{\gamma_1+2} + \tilde{C}_{11} \|X_t^d\|_{\mathcal{F}_0}^{\gamma_1} + C\theta^2 \|X_t^d\|_{\mathcal{F}_0}^{2\alpha_0+2} \right. \\
&\left. + (2\epsilon - 1)\theta^2 \|X_t^d\|_{\mathcal{F}_0}^{2\alpha_0+4} \right) dt + dM_t - \frac{\epsilon}{2} d\langle M \rangle_t.
\end{aligned}$$

We want $2\epsilon - 1 < 0$ and for this reason we choose $\epsilon < \frac{1}{2}$. We know by assumption also that $2\alpha_0 > \gamma_1$. Therefore we control any possible blow-up of the drift term, since there exists a positive constant R such that

$$\tilde{C}_1 \|X_t^d\|_{\mathcal{F}_0}^{\gamma_1+2} + \tilde{C}_{11} \|X_t^d\|_{\mathcal{F}_0}^{\gamma_1} + C\theta^2 \|X_t^d\|_{\mathcal{F}_0}^{2\alpha_0+2} + (2\epsilon-1)\theta^2 \|X_t^d\|_{\mathcal{F}_0}^{2\alpha_0+4} \leq R.$$

Then, as the term on the left hand side has negative leading coefficient, we eventually obtain

$$d \log \left(C + \|X_t^d\|_{\mathcal{F}_0}^2 \right) \leq k_0 + kt + E(\epsilon)$$

with

$$E(\epsilon) := \sup_{t \geq 0} \left(M_t - \frac{\epsilon}{2} \langle M \rangle_t \right).$$

Therefore

$$\sup_{t \in [0, T]} \|X_t^d\|_{\mathcal{F}_0}^2 \leq \tilde{E}(\epsilon)$$

where

$$\tilde{E}(\epsilon) = \exp(k_0 + kT + E(\epsilon)) - C.$$

Using one of the auxiliary results we deduce that

$$\mathbb{P} \left(\sup_{t \in [0, T]} \|X_t^d\|_{\mathcal{F}_0}^2 \geq \beta_1 \right) < \delta_1.$$

Note that the random variable on the right hand side is finite, but not integrable. It follows that

$$\mathbb{P} \left(\sup_{t \in [0, T]} \left(C + \|X_t^d\|_{\mathcal{F}_0} \right) > K \right) \leq \mathbb{P} (E(\epsilon) > \log K - k_0 + kT).$$

Note also that the quantity on the right hand side of the above inequality is independent of the dimension d , (as $E(\epsilon)$ is d -independent and has an exponential distribution), hence

$$\lim_{K \rightarrow \infty} \sup_d \mathbb{P} \left(\sup_{t \in [0, T]} \|X_t^d\|_{\mathcal{F}_0} > K \right) \leq \lim_{K \rightarrow \infty} \mathbb{P} (E(\epsilon) > \log K - k_0 + kT) = 0.$$

Now we can proceed with controlling $\tilde{C}_2 \int_0^t \|X_s^d\|_{\mathcal{F}_1}^{\gamma_2} ds$. We have

$$\tilde{C}_2 \int_0^t \|X_s^d\|_{\mathcal{F}_1}^{\gamma_2} \leq k_0 - \|X_t^d\|_{\mathcal{F}_0}^2 + \int_0^t G\left(\|X_s^d\|_{\mathcal{F}_0}\right) ds + \int_0^t \tilde{B}_s^d(X_s^d) dW_s.$$

Let

$$H_t\left(\|X_t^d\|_{\mathcal{F}_0}\right) := k_0 - \|X_t^d\|_{\mathcal{F}_0}^2 + \int_0^t G\left(\|X_s^d\|_{\mathcal{F}_0}\right) ds$$

and

$$\tilde{\tilde{B}}_t := \int_0^t \tilde{B}_s^d(X_s^d) dW_s.$$

Then

$$\begin{aligned}\tilde{C}_2 \int_0^t \|X_s^d\|_{\mathcal{F}_1}^2 ds &\leq H_t \left(\|X_t^d\|_{\mathcal{F}_0} \right) + \frac{\epsilon}{2} \langle \tilde{B} \rangle_t + \tilde{E}(\epsilon) \\ &\leq C(T, \epsilon) \left(\sup_{t \in [0, T]} \|X_t^d\|_{\mathcal{F}_0}^{q'} \right) + \tilde{E}(\epsilon)\end{aligned}$$

with $\tilde{E}(\epsilon) := \sup_{t \geq 0} \left(\tilde{B}_t - \frac{\epsilon}{2} \langle \tilde{B} \rangle_t \right)$ and q' suitably chosen. This can now be handled as before, which will eventually give

$$\mathbb{P} \left(\int_0^t \|X_s^d\|_{\mathcal{F}_1}^{\gamma_2} ds \geq \beta_2 \right) < \delta_2$$

with $\tilde{C}_2, \gamma_2 > 0$ which gives the required control in Case I.

Steps of the main proof

- ▶ The sequence $(X^d)_d$ is tight in $\mathcal{G}$, and therefore it has a weakly (in distribution) convergent subsequence which we re-index by $(X^d)_d$. By the Skorokhod representation theorem, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ on which the processes $\tilde{X}^d$ are defined and such that they have the same distribution as X^d and $\tilde{X}^d$ converges to a process $\tilde{X}$, $\tilde{\mathbb{P}}$ - almost surely. The sequence $(\tilde{X}^d)_d$ inherits all the distributional properties from the sequence $(X^d)_d$.
- ▶ By the Fatou lemma also the limiting process $\tilde{X}$ satisfies the same controls.
- ▶ On $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ there exists a sequence of Brownian motions $(\tilde{W}_t^d)_d$ such that

$$\tilde{X}_t^d = \tilde{X}_0^d + \int_0^t \tilde{A}_s^d(\tilde{X}_s^d) ds + \int_0^t \theta \|\tilde{X}_s^d\|_{\mathcal{H}_i}^{\alpha_i} \tilde{X}_s^d d\tilde{W}_s^d$$

on $[0, T]$, with $i \in \{0, 1\}$.

Steps of the main proof

- ▶ The sequence $(\tilde{X}^d, \tilde{B}(\tilde{X}), \tilde{W}^d)_d$ is tight in $\mathcal{G} \times \mathcal{G} \times \mathbb{R}$ from which we can deduce that there exists a subsequence (keep the same index) such that

$$\left(\tilde{X}^d, \tilde{B}^d(\tilde{X}^d), \tilde{W}^d, \int_0^\cdot \tilde{B}^d(\tilde{X}_s^d) \tilde{W}_s^d \right)_d$$

converges in distribution to

$$\left(\tilde{X}, \tilde{B}(\tilde{X}), \tilde{W}, \int_0^\cdot \tilde{B}(\tilde{X}_s) \tilde{W}_s \right)_d,$$

where $\tilde{W}$ is a Brownian motion.

- ▶ We pass to the limit in $\mathcal{G}$ in each term of the equation for $\tilde{X}^d$.

Steps of the main proof

- ▶ The convergence holds immediately for the terms $\tilde{X}_t^d$ and $\tilde{X}_0^d$.
- ▶ We show that also both $\int_0^t \tilde{A}_s^d(\tilde{X}_s^d)ds$ and $\int_0^t \tilde{B}_s^d(\tilde{X}_s^d)d\tilde{W}_s^d$ converge to $\int_0^t \tilde{A}(\tilde{X})ds$ and $\int_0^t \tilde{B}(\tilde{X})d\tilde{W}$, respectively, **in** $\mathcal{G}$.
- ▶ For the first integral we use that

$$\|\tilde{A}^d(\tilde{X}^d) - \tilde{A}(\tilde{X})\|_{\mathcal{G}} \leq \|\tilde{A}^d(\tilde{X}^d) - \tilde{A}^d(\tilde{X})\|_{\mathcal{G}} + \|\tilde{A}^d(\tilde{X}) - \tilde{A}(\tilde{X})\|_{\mathcal{G}}.$$

- ▶ The control of each of these terms depends on the initial assumptions, which are different in each of the three cases. We need, therefore, to treat each of these cases separately.

Steps of the main proof

- For showing uniqueness we would need a locally Lipschitz condition such as

$$\|A^d(a) - A^d(b)\|_{\mathcal{G}} \leq C(\|a\|_{\mathcal{F}_1}, \|b\|_{\mathcal{F}_1})\|a - b\|_{\mathcal{G}} \quad (20)$$

where C is continuous as a function of a and b and it is independent from the dimension d . This implies uniqueness of the maximal solution (which we have already assumed) and then one can deduce the uniqueness of the newly-constructed global solution: we know that any maximal solution $\hat{X}$ is unique up to a maximal stopping time $\mathcal{T}$. That is $\hat{X} = X$ on $[0, \mathcal{T})$. However, we show that X itself is a **global** solution. As a consequence, we need to have $\mathbb{P}(\mathcal{T} = \infty) = 1$.

- Note also that condition (20) is a generalised version of (8) and (15), so assumptions (A1) – (A3) imply directly uniqueness for the maximal solution.

Final remarks

- ▶ A large class of PDEs exists in the literature for which global existence of solutions does not necessarily hold, with several such examples coming from geophysical fluid dynamics.
- ▶ We show that the addition of a suitable stochastic term to this type of PDEs ensures existence of a global solution. The stochastic term is not a martingale but just a local martingale, so all uniform controls, tightness and convergence results have to be adapted to cope with this.
- ▶ The class of models includes:
 - ▶ Viscous compressible and incompressible models e.g. incompressible Navier Stokes equation in vorticity form on the 3D torus, viscous 2D rotating shallow water model
 - ▶ Inviscid compressible models e.g. Burgers' equation on the 1D/2D torus, 2D inviscid rotating shallow water model.
 - ▶ Inviscid incompressible models e.g. incompressible Euler equation in vorticity form the 3D torus.

Thank you!