

A regularized truncated finite element method for a degenerate parabolic stochastic PDE on non-compact graphs

Mihály Kovács

(joint work with Jianbo Cui and Derui Sheng)

Pázmány Péter Catholic University and Budapest University of Technology and Economics
and Chalmers University

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Introduction

- Consider a graph Γ consisting of a finite set of vertices $\{O_i\}_{i=1}^n$ and a finite set of edges $\{I_k\}_{k=1}^m$
- Each edge I_k connects a pair of vertices and is identified with an interval $[a_k, b_k]$ with $-\infty < a_k \leq b_k \leq \infty$
- A location $(z, k) \in \Gamma$ on the graph represents a position $z \in [a_k, b_k]$ on the edge I_k
- Consider the following SPDE on the graph Γ :

$$\partial_t u(t, z, k) = \mathcal{L}u(t, z, k) + b(u(t, z, k)) + g(u(t, z, k))\partial_t W(t, z, k), \quad (1)$$

$t \in (0, T]$ and $(z, k) \in \Gamma$ with initial condition $u(0, z, k) = u_0(z, k)$.

- Here $\mathcal{L}$ is a second order differential operator:

$$\mathcal{L}f(z, k) = \frac{1}{2\beta_k(z)} \frac{d}{dz} \left(\alpha_k(z) \frac{df}{dz}(z, k) \right), \quad \text{for } z \in (a_k, b_k), \quad (2)$$

$k = 1, 2, \dots, m$ with Kirchhoff-type gluing conditions at interior vertices

- W is a Q -Wiener process on an appropriate Hilbert space on Γ

Introduction: motivation

- $-H : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a stream function that describes the flow pattern of a fluid or gas in two-dimensional space
- $\mathcal{W}$ is a spatially homogeneous Wiener process on $\mathbb{R}^2$ with finite spectral measure μ
- For $\varepsilon > 0$, consider a stochastic PDE

$$\partial_t u_\varepsilon(t, x) = \mathcal{L}_\varepsilon u_\varepsilon(t, x) + b(u_\varepsilon(t, x)) + g(u_\varepsilon(t, x)) \partial_t \mathcal{W}(t, x), \quad (3)$$

subject to a suitable initial condition, H smooth with appropriate additional assumptions and $b, g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be Lipschitz continuous

- The linear operator $\mathcal{L}_\varepsilon = \frac{1}{2}\Delta + \frac{1}{\varepsilon}\nabla^\perp H \cdot \nabla$ is the infinitesimal generator of the diffusion process

$$dX_\varepsilon(t) = \frac{1}{\varepsilon} \nabla^\perp H(X_\varepsilon(t)) dt + dB(t), \quad t \in [0, T],$$

where B is a two dimensional Brownian motion, and

$$\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$$

Introduction: motivation

- Consider $C_k(z)$, the connected components of a given level set $C(z) := \{x \in \mathbb{R}^2 : H(x) = z\}$
- By identifying all points in $\mathbb{R}^2$ in the same connected component $C_k(z)$, we obtain a graph Γ consisting of a finite set of vertices $\{O_i\}_{i=1}^n$ and a finite set of edges $\{I_k\}_{k=1}^m$
- Each vertex O_i corresponds to a critical point $\mathbf{x}_i$ of the function H , where interior vertices correspond to saddle points of H , and exterior vertices correspond to local extremum points of H
- Each interior and exterior vertex is incident to three edges and a single edge, respectively
- We also include O_∞ among exterior vertices, the endpoint of the only unbounded interval I_m on the graph

Introduction: motivation

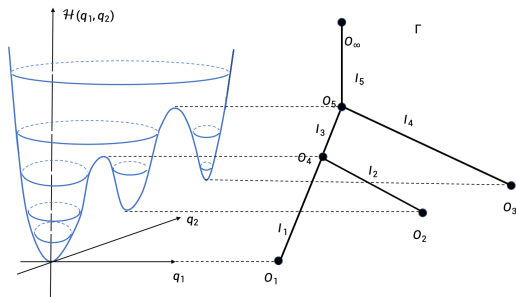


Figure: An example for the Hamiltonian H with three local minimizers (O_1 , O_2 , and O_3) and two saddle points (O_4 and O_5), and the graph Γ .

Introduction: motivation

- Denote by $\Pi : \mathbb{R}^2 \rightarrow \Gamma$ the identification map associating every $x \in \mathbb{R}^2$ to the corresponding point $\Pi(x)$ on the graph Γ .
- Then $\Pi(x) = (H(x), k(x))$, where $k(x) \in \{1, 2, \dots, m\}$ denotes the index of the edge $l_{k(x)}$ to which $\Pi(x)$ belongs.
- we have $l_m \cong [H_0, \infty]$, where $H_0 := \max_{1 \leq i \leq n} H(\mathbf{x}_i)$
- For $1 \leq k \leq m-1$, $l_k \cong [H(\mathbf{x}_i), H(\mathbf{x}_j)]$, where O_i and O_j are the two vertices incident to l_k ordered so that $H(\mathbf{x}_i) < H(\mathbf{x}_j)$.
- It was shown by Cerrai and Freidlin in [3] that for every $p \geq 1$ and $0 < \tau < T$ one has

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \sup_{t \in [\tau, T]} \|u_\varepsilon(t) - u(t) \circ \Pi\|^p = 0,$$

in a weighted L^2 -space, where u is the solution of (1) with coefficients α_k and β_k are defined through the Hamiltonian H as

$$\alpha_k(z) = \oint_{C_k(z)} |\nabla H(x)| dl_{z,k}, \quad \beta_k(z) = \oint_{C_k(z)} \frac{1}{|\nabla H(x)|} dl_{z,k},$$

where $dl_{z,k}$ is the length element on $C_k(z)$

We aim to approximate the solution of (1). The major obstacles are:

- The graph is non-compact, the m th edge, identified by I_m is unbounded
- It is known that: β_k and α_k are continuously differentiable in the interior $\overset{\circ}{I}_k$ of I_k . However, if (z, k) approaches an endpoint of an edge I_k , corresponding to a vertex $O_i = (H(\mathbf{x}_i), k)$, then, as $(z, k) \rightarrow O_i$,

$$\alpha_k(z) \sim \begin{cases} \text{const} \cdot |z - H(\mathbf{x}_i)|, & \text{if } \mathbf{x}_i \text{ is a local extremum point,} \\ \text{const}, & \text{if } \mathbf{x}_i \text{ is a saddle point,} \\ \text{const} \cdot z, & \text{if } O_i = O_\infty; \end{cases}$$
$$\beta_k(z) \sim \begin{cases} \text{const}, & \text{if } \mathbf{x}_i \text{ is a local extremum point,} \\ \text{const} \cdot |\log |z - H(\mathbf{x}_i)||, & \text{if } \mathbf{x}_i \text{ is a saddle point,} \\ \text{const}, & \text{if } O_i = O_\infty. \end{cases}$$

This talk focuses on spatial approximation.

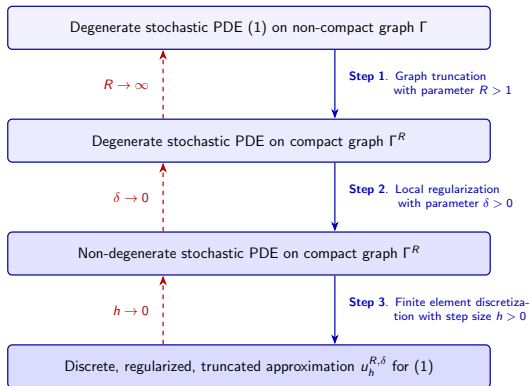


Figure: The three-step numerical strategy for approximating the solution to (1).

Notation

- Let $\mu = \{\mu_k\}_{k=1}^m$ and $\nu = \{\nu_k\}_{k=1}^m$ be two weight functions on Γ , that is, $\mu_k, \nu_k : I_k \rightarrow [0, \infty)$ are measurable functions for each $k = 1, 2, \dots, m$
- For any subinterval $J \subset I_k$, we define $L_{\nu_k}^2(J)$ as the space of complex-valued functions on J equipped with the inner product

$$\langle w, v \rangle_{L_{\nu_k}^2(J)} := \int_J w(z) \overline{v(z)} \nu_k(z) dz, \quad w, v \in L_{\nu_k}^2(J),$$

where $\overline{v(z)}$ denotes the complex conjugate of $v(z)$

- The weighted Sobolev space $W_{\mu_k, \nu_k}^{1,2}(J)$ consists of all functions $w \in L_{\nu_k}^2(J)$ whose weak derivative $w' := \frac{d}{dz} w \in L_{\mu_k}^2(J)$, endowed with the inner product

$$\langle w, v \rangle_{W_{\mu_k, \nu_k}^{1,2}(J)} := \langle w, v \rangle_{L_{\nu_k}^2(J)} + \langle w', v' \rangle_{L_{\mu_k}^2(J)}, \quad w, v \in W_{\mu_k, \nu_k}^{1,2}(J)$$

- Graph-weighted spaces are direct sums

$$L_{\nu}^2(\Gamma) := \bigoplus_{k=1}^m L_{\nu_k}^2(I_k), \quad W_{\mu, \nu}^{1,2}(\Gamma) := \bigoplus_{k=1}^m W_{\mu_k, \nu_k}^{1,2}(I_k).$$

In the special case $\mu = \nu \equiv 1$, we omit the weights

- A function f defined on the graph Γ is said to be continuous at a vertex O_i if for all edges $\{l_k\}$ incident to O_i , the corresponding coordinate functions $\{f(\cdot, k)\}$ attain the same value at O_i
- Denote by $W_{\mu,\nu}^{1,2,c}(\Gamma)$ the subspace of $W_{\mu,\nu}^{1,2}(\Gamma)$ consisting of functions that are continuous at each interior vertex on the graph Γ
- A function f of Γ satisfies the Kirchhoff condition if

$$\sum_{k:l_k \sim O_i} \alpha_k(H(\mathbf{x}_i)) d_k f(H(\mathbf{x}_i), k) = 0 \quad \text{for each interior vertex } O_i,$$

where $d_k f(H(\mathbf{x}_i), k) := \Phi_{k,i} \frac{d}{dz} f(H(\mathbf{x}_i), k)$ with

$$\Phi_{k,i} = \begin{cases} +1, & \text{if } l_k \text{ is incident to } O_i \text{ and } z \text{ decreases along } l_k \text{ towards } O_i, \\ -1, & \text{if } l_k \text{ is incident to } O_i \text{ and } z \text{ increases along } l_k \text{ towards } O_i. \end{cases}$$

Operators and semigroups

- Recall that $\mathcal{L}$ is a second order differential operator:

$$\mathcal{L}f(z, k) = \frac{1}{2\beta_k(z)} \frac{d}{dz} \left(\alpha_k(z) \frac{df}{dz}(z, k) \right), \quad \text{for } z \in (a_k, b_k), \quad (4)$$

- Let $\gamma : \Gamma \rightarrow (0, \infty)$, $(z, k) \mapsto \gamma_k(z)$, be continuous on Γ and continuously differentiable on the interior of each edge. Suppose further that

$$|\gamma'_k(z)| \leq C, \text{ if } (z, k) \in \dot{I}_k, 1 \leq k \leq m-1;$$

and

$$\sqrt{z}|\gamma'_m(z)| \leq C\gamma_m(z), \text{ if } (z, m) \in \dot{I}_m,$$

where $C > 0$ is a constant independent of (z, k) .

Remark. One can show that there exists a constant $\kappa > 0$ such that

$$\alpha_k(z)|\gamma'_k(z)|^2 \leq \kappa\beta_k(z)\gamma_k^2(z) \quad \text{for } (z, k) \in \dot{I}_k, k = 1, \dots, m.$$

Example. For $(z, k) \in \Gamma$ with $k = 1, \dots, m-1$, set $\gamma_k(z) = 1$. On the unbounded edge $I_m \cong [H_0, \infty)$, where $H_0 = \max_{1 \leq i \leq n} H(\mathbf{x}_i)$, we consider the following two choices:

$$(i) \quad \gamma_m(z) = e^{-\rho_1(z-H_0)^{\rho_2}}, \quad (ii) \quad \gamma_m(z) = (z - H_0 + 1)^{-\rho_3},$$

where $\rho_1 > 0, 0 < \rho_2 < \frac{1}{2}$ and $\rho_3 > 1$.

Operators and semigroups

Theorem. For any $\lambda \geq \frac{1}{8}\kappa$, the operator $-\lambda I + \mathcal{L}$ generates an analytic contraction semigroup on $L^2_{\beta\gamma}(\Gamma)$, with domain $D(\mathcal{L})$ of $\mathcal{L}$ is given by

$$D(\mathcal{L}) = \left\{ f \in W^{1,2,c}_{\alpha\gamma,\beta\gamma}(\Gamma) : f \text{ satisfies the Kirchoff condition, } \alpha_k \frac{df}{dz} \text{ is weakly differentiable on each } I_k, \text{ and } \mathcal{L}f \in L^2_{\beta\gamma}(\Gamma) \right\}.$$

- When $1_\Gamma \in L^2_{\beta\gamma}(\Gamma)$, then W can be regarded as a $L^2_{\beta\gamma}(\Gamma)$ -valued nuclear Wiener process.
- Let B and G be the Nemytskii operators associated with b and g , respectively, i.e., for $f \in L^2_{\beta\gamma}(\Gamma)$ and $\psi \in \mathcal{U}_0$,

$$B(f)(z, k) := b(f(z, k)), \quad G(f)(\psi)(z, k) := g(f(z, k))\psi(z, k),$$

for $(z, k) \in \Gamma$.

- We are looking for a *mild solution* of (1): i.e., for any $t \in [0, T]$,

$$u(t) = e^{t\mathcal{L}} u_0 + \int_0^t e^{(t-s)\mathcal{L}} B(u(s)) ds + \int_0^t e^{(t-s)\mathcal{L}} G(u(s)) dW(s).$$

Theorem. Suppose that $\{1_\Gamma, u_0\} \subset L^2_{\beta\gamma}(\Gamma)$. Then there exists a unique mild solution $u = \{u(t)\}_{t \in [0, T]}$ to (1). Moreover, for any $p \geq 1$, there exists $C := C(p, \kappa, T) > 0$ such that

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|u(t)\|_{L^2_{\beta\gamma}(\Gamma)}^p \right] \leq C \left(1 + \|u_0\|_{L^2_{\beta\gamma}(\Gamma)}^p \right).$$

and that

$$\mathbb{E} \int_0^T \sum_{k=1}^m \int_{I_k} \left| \frac{\partial}{\partial z} u(t, z, k) \right|^2 \alpha_k(z) \gamma_k(z) dz dt \leq C \left(1 + \|u_0\|_{L^2_{\beta\gamma}(\Gamma)}^2 \right).$$

Truncated approximation

- For each $R > 0$, let $\eta_R : \Gamma \rightarrow [0, 1]$ be a continuous cut-off function such that

$$\eta_R(z, k) = \begin{cases} 1, & \text{if } 0 \leq z \leq R, \\ 0, & \text{if } z \geq R + 1. \end{cases}$$

- Since η_R does not depend on the edge index k , we write $\eta_R(z)$ in place of $\eta_R(z, k)$ for simplicity.
- Suppose that $R > H_0$ so that $1 - \eta_R$ vanishes identically on $\cup_{k=1}^{m-1} I_k$.
- Consider following truncated problem

$$\begin{aligned} \partial_t u^R(t, z, k) = \mathcal{L}^R u^R(t, z, k) + b(u^R(t, z, k))\eta_R(z) \\ + g(u^R(t, z, k))\eta_R(z)\partial_t W(t, z, k) \end{aligned}$$

subject to the initial value $u^R(0) = u_0$, where

$$\mathcal{L}^R f(z, k) = \frac{1}{2\beta_k(z)} \frac{d}{dz} \left(\alpha_k^R(z) \frac{df}{dz}(z, k) \right), \quad \text{if } (z, k) \in \dot{I}_k,$$

$k = 1, \dots, m$, with $\alpha_k^R := \alpha_k \cdot \eta_R$ and supplemented by a Kirchoff gluing condition.

Truncated approximation

Theorem. Suppose that $\{1_\Gamma, u_0\} \subset L^2_{\beta\gamma}(\Gamma) \cap L^2_{\beta\sqrt{\gamma}}(\Gamma)$. Then there exists a constant $C := C(T, u_0) > 0$ such that for any $t \in [0, T]$ and $R \geq H_0 + 1$,

$$\mathbb{E} \left[\|u(t) - u^R(t)\|_{L^2_{\beta\gamma}(\Gamma)}^2 \right] \leq C \sup_{z \geq R-1} \left\{ \alpha_m(z) \sqrt{\gamma_m(z)} \right\}.$$

Remark. We have that

$$1_\Gamma \subset L^2_{\beta\gamma}(\Gamma) \cap L^2_{\beta\sqrt{\gamma}}(\Gamma), \quad \text{and} \quad \lim_{R \rightarrow \infty} \sup_{z \geq R} z \sqrt{\gamma_m(z)} = 0.$$

holds for the graph weight function γ in both cases (i) and (ii) of our Example provided that $\rho_3 > 2$.

Remark. As $\text{supp}(\eta_R) \subset [0, R+1]$, we have $\partial_t u^R(t, z, m) = 0$ for any $z \geq R+1$ and thus $u^R(t, z, m) = u^R(0, z, m) = u_0(z, m)$ for all $t \in [0, T]$ and $z \geq R+1$. Thus, it suffices to consider the compact graph

$$\Gamma^R := \{(z, k) \in \Gamma : z \leq R+1\}.$$

It consists of m edges $\{J_k\}_{k=1}^m$ and $n+1$ vertices $\{O_i\}_{i=1}^{n+1}$ with additional vertex $O_{n+1} := (R+1, m)$. For $k = 1, \dots, m-1$, the edges are given by $J_k = I_k$, with the last edge $J_m := \{(z, m) \in I_m : z \leq R+1\} \cong [H_0, R+1]$

Truncated approximation

- We thus consider a differential operator $\check{\mathcal{L}}$ on Γ^R as

$$\check{\mathcal{L}}f(z, k) = \frac{1}{2\beta_k(z)} \frac{d}{dz} \left(\alpha_k^R(z) \frac{d}{dz} f(z, k) \right), \quad \text{if } (z, k) \in \mathring{J}_k, \quad k = 1, \dots, m$$

endowed with the same Kirchoff gluing conditions as $\mathcal{L}^R$.

- In order to reproduce the same asymptotic behaviour of α^R at $O_{n+1} := (R+1, m)$ as at other exterior vertices we choose the cut-off function η_R such that, for some $K_0 \neq 0$,

$$\frac{d^-}{dz} \eta_R(R+1) = \lim_{z \rightarrow (R+1)^-} \frac{\eta_R(z) - \eta_R(R+1)}{z - (R+1)} = K_0$$

- Restricted to the compact graph Γ^R , the stochastic process $\check{u}(t) := u^R(t) |_{\Gamma^R}$ satisfies

$$d\check{u}(t) = (\check{\mathcal{L}}\check{u}(t) + \check{B}(\check{u}(t)))dt + \check{G}(\check{u}(t))dW(t), \quad \check{u}(0) = u_0 |_{\Gamma^R}.$$

with

$$\check{B}(f_1)(z, k) := b(f_1(z, k))\eta_R(z), \quad \check{G}(f_1)(\psi)(z, k) = g(f_1(z, k))\eta_R(z)\psi(z, k),$$

for $(z, k) \in \Gamma^R$.

Regularized truncated approximation

Problem: the operator $\tilde{\mathcal{L}}$ fails to be uniformly elliptic on the graph Γ^R

Regularization:

- It suffices to regularize α_k^R near exterior vertices and β_k near interior vertices
- For each edge J_k with $k = 1, 2, \dots, m-1$ on Γ^R , we divide it into two subintervals $J_k^{(1)}$ and $J_k^{(2)}$ of equal length. For $k = m$, we split $J_m \cong [H_0, R+1]$ into two subintervals

$$J_m^{(1)} \cong [H_0, H_0 + 1] \quad \text{and} \quad J_m^{(2)} \cong [H_0 + 1, R + 1]$$

- With this decomposition, each subinterval $\{J_k^{(l)}\}_{l=1,2}$ is incident to exactly one vertex of the compact graph Γ^R
- Set $\delta_{\min} := \frac{1}{4} \min_{1 \leq k \leq m} |J_k|$, where $|J_k|$ is the length of the edge J_k . For $\delta \in (0, \delta_{\min})$, we define

$$\begin{aligned} \beta_k^\delta(z) &:= \beta_k(z), \quad (z, k) \in J_k^{(l)}, \text{ if } J_k^{(l)} \text{ is incident to an exterior vertex } O_i, \\ \alpha_k^{R,\delta}(z) &:= \alpha_k^R(z), \quad (z, k) \in J_k^{(l)}, \text{ if } J_k^{(l)} \text{ is incident to an interior vertex } O_i \end{aligned}$$

where $O_i = (H(\mathbf{x}_i), k)$

Regularized truncated approximation

- For the remaining cases we regularize under the following assumptions: the function $\alpha_k^{R,\delta}$ is continuously differentiable on the edge $\mathring{J}_k$ and for some $\delta_0 > 0$ and for all $\delta \in (0, \delta_0)$, there exist positive constants $\mathfrak{c}_1(\delta)$ and $\mathfrak{c}_4(\delta)$ depending on δ such that for any $(z, k) \in \mathring{J}_k$ and $k = 1, \dots, m$,

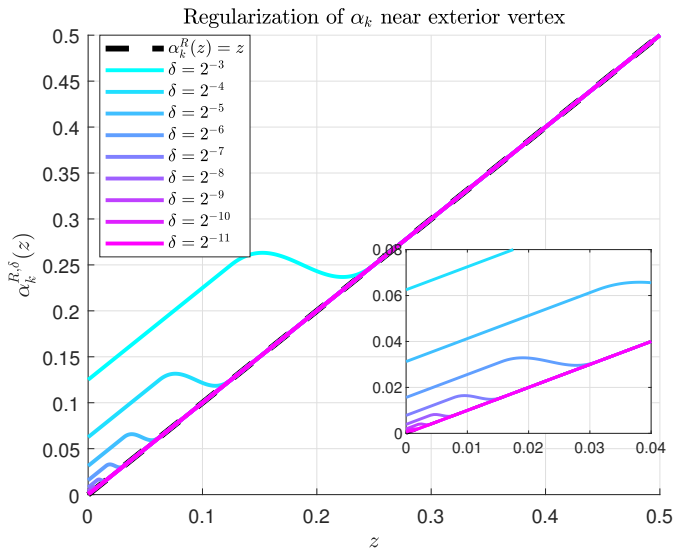
$$\max\{\mathfrak{c}_0 \alpha_k^R(z), \mathfrak{c}_1(\delta)\} \leq \alpha_k^{R,\delta}(z) \leq \mathfrak{c}_2(\alpha_k(z) + 1), \quad \lim_{\delta \rightarrow 0} \alpha_k^{R,\delta}(z) = \alpha_k^R(z),$$

$$0 < \mathfrak{c}_3 \leq \beta_k^\delta(z) \leq \min\{\mathfrak{c}_4(\delta), \mathfrak{c}_5 \beta_k(z)\}, \quad \lim_{\delta \rightarrow 0} \beta_k^\delta(z) = \beta_k(z),$$

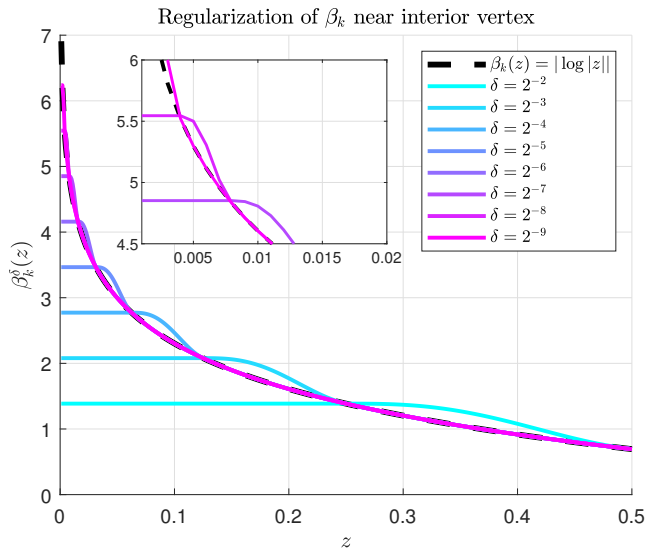
where $\mathfrak{c}_0, \mathfrak{c}_2, \mathfrak{c}_3$, and $\mathfrak{c}_5$ are some constants independent of R and δ .

Remark. This always can be achieved.

Regularized truncated approximation



Regularized truncated approximation



Regularized truncated approximation

- For $\delta \in (0, \delta_0)$, we define the regularization of $\check{\mathcal{L}}$ as follows

$$\check{\mathcal{L}}^\delta f(z, k) = \frac{1}{2\beta_k^\delta(z)} \frac{d}{dz} \left(\alpha_k^{R, \delta}(z) \frac{d}{dz} f(z, k) \right), \text{ if } (z, k) \in J_k, k = 1, \dots, m,$$

endowed with Kirchoff gluing conditions at each vertex O_i on Γ^R

- Consider the regularized truncated approximation of (1):

$$d\check{u}^\delta(t) = (\check{\mathcal{L}}^\delta \check{u}^\delta(t) + \check{B}(\check{u}^\delta(t)))dt + \check{G}(\check{u}^\delta(t))dW(t), \quad \check{u}^\delta(0) = \check{u}(0)$$

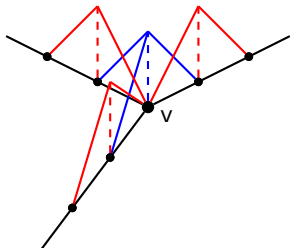
Theorem. Under the above assumptions on the regularization, if $1_\Gamma \in L^2_{\beta\gamma}(\Gamma)$, and $u_0 \in W^{1,2}_{(\alpha+1)\gamma, \beta\gamma}(\Gamma)$, then for any $R > H_0$, $p \geq 1$ and $t \in [0, T]$,

$$\lim_{\delta \rightarrow 0} \mathbb{E} \left[\sup_{t \in [0, T]} \|\check{u}^\delta(t) - \check{u}(t)\|_{L^2_{\beta\gamma}(\Gamma^R)}^p \right] = 0.$$

Regularized truncated finite element approximation

- Subdivide each edge J_k into $n_k \geq 2$ intervals of length h_k
- Standard hat functions on the edges, $V_h^k = \text{span}\{\psi_j^k\}_{j=1,2,\dots,n_k-1}$
- Special hat functions on the vertices O_i , $i = 1, \dots, n+1$, $\{\phi_i\}_{i=1}^{n+1}$
- Finite element space is given by

$$\mathbb{V}_h(\Gamma^R) = \left(\bigoplus_{k=1}^m V_h^k \right) \oplus \text{span}\{\phi_i\}_{i=1}^{n+1}$$



Regularized truncated approximation

- Denote by $W_{\text{cont}}^{1,2}(\Gamma^R) := W_{\alpha^R, \delta_\gamma, \beta^\delta \gamma}^{1,2,c}(\Gamma^R)$ the subspace of $W_{\alpha^R, \delta_\gamma, \beta^\delta \gamma}^{1,2}(\Gamma^R)$ consisting of functions that are continuous at each interior vertex of Γ^R
- For $\lambda \in \mathbb{R}$, consider the bilinear form $a_\lambda^{R,\delta} : W_{\text{cont}}^{1,2}(\Gamma^R) \times W_{\text{cont}}^{1,2}(\Gamma^R) \rightarrow \mathbb{C}$ defined by

$$a_\lambda^{R,\delta}(\phi, \psi) = \lambda \sum_{k=1}^m \int_{J_k} \phi \bar{\psi} \beta_k^\delta \gamma_k dz + \frac{1}{2} \sum_{k=1}^m \int_{J_k} \alpha_k^{R,\delta} \frac{d}{dz} \phi \frac{d}{dz} (\bar{\psi} \gamma_k) dz,$$

$$\phi, \psi \in W_{\text{cont}}^{1,2}(\Gamma^R)$$

- Since $\mathbb{V}_h(\Gamma^R) \subset W_{\text{cont}}^{1,2}(\Gamma^R)$, we can define the discrete operator $\check{\mathcal{L}}_h^\delta : \mathbb{V}_h(\Gamma^R) \rightarrow \mathbb{V}_h(\Gamma^R)$ by

$$-\langle \check{\mathcal{L}}_h^\delta \varphi, \psi \rangle_{L_{\beta^\delta \gamma}^2(\Gamma^R)} = a_0^{R,\delta}(\varphi, \psi), \quad \varphi, \psi \in \mathbb{V}_h(\Gamma^R)$$

- Define the $L_{\beta^\delta \gamma}^2(\Gamma^R)$ -projection $\mathcal{P}_h$ defined, respectively, via

$$\langle \mathcal{P}_h \varphi, \psi \rangle_{L_{\beta^\delta \gamma}^2(\Gamma^R)} = \langle \varphi, \psi \rangle_{L_{\beta^\delta \gamma}^2(\Gamma^R)}, \quad \varphi \in L_{\beta^\delta \gamma}^2(\Gamma^R), \psi \in \mathbb{V}_h(\Gamma^R)$$

Regularized truncated finite element approximation

Consider the following spatial semi-discrete approximation of the truncated regularized problem:

$$d\check{u}_h^\delta(t) = (\mathcal{L}_h^\delta \check{u}_h^\delta(t) + \mathcal{P}_h \check{B}(\check{u}_h^\delta(t)))dt + \mathcal{P}_h \check{G}(\check{u}_h^\delta(t))dW(t), \quad \check{u}_h^\delta(0) = \mathcal{P}_h \check{u}(0).$$

Theorem. Assume further to the above that for any fixed $R > H_0$, $\delta \in (0, \delta_0)$, and $k = 1, \dots, m$, $(\alpha_k^{R,\delta})' \in L^4(J_k)$. If $\{u_0, 1_\Gamma\} \subset L^2_{\beta\gamma}(\Gamma)$, then for any $R \geq H_0$, $\delta \in (0, \delta_0)$, and $\vartheta \in (0, \frac{1}{2})$, there exists $C := C(R, \delta, \vartheta, u_0) > 0$ such that

$$\mathbb{E} \left[\|\check{u}^\delta(t) - \check{u}_h^\delta(t)\|_{L^2_{\beta\gamma}(\Gamma^R)}^2 \right] \leq Ch^{2\vartheta}(1 + t^{-\vartheta}).$$

Finally, for sufficiently large $R > H_0$ and small $\delta, h > 0$, we define the discrete regularized truncated approximation

$$u_h^{R,\delta}(t, z, k) := \begin{cases} \check{u}_h^\delta(t, z, k), & \text{if } (z, k) \in \Gamma^R, \\ u_0(z, k), & \text{if } (z, k) \in \Gamma - \Gamma^R \end{cases}$$

for approximating the mild solution $u(t, z, k)$ to the stochastic PDE (1) on the graph Γ .

A convergence result

Corollary. Assume the above conditions on the regularization and the cut-off function η_R and suppose that

$$1_\Gamma \subset L^2_{\beta\gamma}(\Gamma) \cap L^2_{\beta\sqrt{\gamma}}(\Gamma), \quad \text{and} \quad \lim_{R \rightarrow \infty} \sup_{z \geq R} z \sqrt{\gamma_m(z)} = 0.$$

If $u_0 \in W^{1,2}_{(\alpha+1)\gamma, \beta\gamma}(\Gamma) \cap L^2_{\beta\sqrt{\gamma}}(\Gamma)$, then for any $t \in (0, T]$,

$$\lim_{R \rightarrow \infty} \lim_{\delta \rightarrow 0} \lim_{h \rightarrow 0} \mathbb{E} \left[\|u(t) - u_h^{R,\delta}(t)\|_{L^2_{\beta\gamma}(\Gamma)}^2 \right] = 0.$$

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