

Drift parameter estimation in CIR and CKLS models

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Joint work and sources

This talk is based on joint work with

Yuliya Mishura, Oksana Chernova, Olena Dehtiar, and Olha Prykhodko.

The results presented here are based on:

- ▶ O. Dehtiar, Yu. Mishura and K. Ralchenko, “Two methods of estimation of the drift parameters of the Cox–Ingersoll–Ross process: Continuous observations,” *Communications in Statistics – Theory and Methods* **51**(19), 6818–6833 (2022).
- ▶ O. Dehtiar, Yu. Mishura and K. Ralchenko, “Parameter estimation in CKLS model by continuous observations,” *Statistics & Probability Letters* **184**, Art. 109391 (2022).
- ▶ O. Chernova, O. Dehtiar, Yu. Mishura and K. Ralchenko, “Rate of convergence of discretized drift parameters estimators in the Cox–Ingersoll–Ross model,” *Communications in Statistics – Theory and Methods* **53**(13), 4857–4879 (2024).
- ▶ O. Prykhodko and K. Ralchenko, “Discretization and asymptotic normality of drift parameter estimator in the Cox–Ingersoll–Ross model,” *Austrian Journal of Statistics* **54**(1), 82–99 (2025).

CIR model

The Cox–Ingersoll–Ross (CIR) process is defined by

$$dr_t = (a - br_t) dt + \sigma \sqrt{r_t} dW_t, \quad r_0 > 0,$$

where $W = \{W_t, t \geq 0\}$ is a Wiener process and $a, b, \sigma > 0$.

Why is this model important?

- ▶ classical positive diffusion model;
- ▶ widely used in short-rate modeling;
- ▶ combines mean reversion with state-dependent volatility.

Goal: estimate the drift parameters (a, b) from continuous and discrete observations.

Two estimation approaches

Maximum likelihood

- ▶ classical likelihood-based method;
- ▶ statistically efficient;
- ▶ involves inverse powers of r_t ;
- ▶ well-defined for $2a \geq \sigma^2$; Gaussian regime: $2a > \sigma^2$.

Moment-based estimator

- ▶ based on ergodic averages;
- ▶ explicit and computationally simple;
- ▶ avoids inverse powers of r_t ;
- ▶ works for all $a, b, \sigma > 0$.

Guiding theme

The talk highlights the trade-off between the efficiency of the MLE and the broader applicability of the moment-based estimator.

CIR process: boundary behavior and ergodicity

Under $a, b, \sigma > 0$, the CIR equation has a unique nonnegative strong solution. The boundary behavior depends on the Feller condition:

$$2a \geq \sigma^2 \implies r_t > 0 \text{ for all } t \geq 0 \text{ a.s.}$$

If $0 < 2a < \sigma^2$, the origin is attainable but instantaneously reflecting.

The process is ergodic with gamma stationary density

$$p_\infty(x) = \frac{\lambda^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\lambda x} \mathbf{1}_{\{x>0\}}, \quad \nu = \frac{2a}{\sigma^2}, \quad \lambda = \frac{2b}{\sigma^2}.$$

Role in estimation

Time averages converge to stationary expectations and identify the parameters.

Key ergodic limits

As $T \rightarrow \infty$, the following convergences hold almost surely:

$$\frac{1}{T} \int_0^T r_t dt \rightarrow \frac{a}{b},$$

$$\frac{1}{T} \int_0^T r_t^2 dt \rightarrow \frac{a(a + \sigma^2/2)}{b^2},$$

$$\frac{1}{T} \int_0^T \frac{dt}{r_t} \rightarrow \frac{2b}{2a - \sigma^2}, \quad \text{if } 2a > \sigma^2.$$

Consequence

The inverse-moment limit explains the restriction for the MLE, while the first and second moment limits motivate the moment-based estimator.



Maximum likelihood estimation

Continuous-time MLE

The likelihood based on continuous observations is

$$\mathcal{L} = \exp \left\{ \int_0^T \frac{a - br_t}{\sigma^2 r_t} dr_t - \frac{1}{2} \int_0^T \frac{(a - br_t)^2}{\sigma^2 r_t} dt \right\}.$$

Maximizing $\mathcal{L}$ with respect to (a, b) gives the explicit estimator

$$\begin{aligned}\hat{a}_T &= \frac{\int_0^T r_t dt \int_0^T \frac{dr_t}{r_t} - T(r_T - r_0)}{\int_0^T r_t dt \int_0^T \frac{dt}{r_t} - T^2}, \\ \hat{b}_T &= \frac{(r_0 - r_T) \int_0^T \frac{dt}{r_t} + T \int_0^T \frac{dr_t}{r_t}}{\int_0^T r_t dt \int_0^T \frac{dt}{r_t} - T^2}.\end{aligned}$$

Recall: this estimator is well defined when $2a \geq \sigma^2$; standard Gaussian asymptotics require $2a > \sigma^2$.

Asymptotic properties of the MLE

Theorem 1

If $2a \geq \sigma^2$, then

$$(\hat{a}_T, \hat{b}_T) \rightarrow (a, b) \quad a.s. \text{ as } T \rightarrow \infty.$$

Theorem 2

If $2a > \sigma^2$, then

$$\sqrt{T} \begin{pmatrix} \hat{a}_T - a \\ \hat{b}_T - b \end{pmatrix} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma_{\text{mle}}),$$

where

$$\Sigma_{\text{mle}} = \begin{pmatrix} \frac{a}{b}(2a - \sigma^2) & 2a - \sigma^2 \\ 2a - \sigma^2 & 2b \end{pmatrix}.$$

Additionally: for $2a > \sigma^2$, we obtain explicit convergence rates in probability.

Discrete observations and discretization

In practice, the process is observed on a high-frequency, long-span grid

$$m_n = \lceil n^{1+\beta} \rceil, \quad \delta_n = \frac{n}{m_n}, \quad t_k = k\delta_n, \quad k = 0, \dots, m_n.$$

Thus $t_{m_n} = n$, $m_n \sim n^{1+\beta}$ and $\delta_n \sim n^{-\beta}$.

The continuous statistics over $[0, n]$ are replaced by Riemann-type sums:

$$\int_0^n r_t dt \Rightarrow \sum r_{t_k} \delta_n, \quad \int_0^n \frac{dt}{r_t} \Rightarrow \sum \frac{\delta_n}{r_{t_k}}, \quad \int_0^n \frac{dr_t}{r_t} \Rightarrow \sum \frac{\Delta r_{t_k}}{r_{t_k}}.$$

Resulting estimator

Substituting these approximations into the MLE gives the discretized estimators $(\hat{a}_n^d, \hat{b}_n^d)$.

Discrete MLE: asymptotic properties

Theorem 3

Assume $a > \sigma^2$. For every $\beta > 0$,

$$(\hat{a}_n^d, \hat{b}_n^d) \rightarrow (a, b) \quad \text{in probability.}$$

If $\beta \geq 2$, the convergence is almost sure.

Theorem 4

Assume that $a > \sigma^2$ and $\beta > 1$. Then

$$\sqrt{n} \begin{pmatrix} \hat{a}_n^d - a \\ \hat{b}_n^d - b \end{pmatrix} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma_{\text{mle}}),$$

where Σ_{mle} is the asymptotic covariance matrix of the continuous-time MLE.



Moment-based estimation approach

Key idea of the moment-based approach

The MLE requires the inverse integral

$$\int_0^T \frac{dt}{r_t},$$

which leads to the restriction $2a \geq \sigma^2$ for well-definedness, and $2a > \sigma^2$ for standard Gaussian asymptotics.

The moment-based construction uses only

$$\frac{1}{T} \int_0^T r_t dt \quad \text{and} \quad \frac{1}{T} \int_0^T r_t^2 dt,$$

whose limits are valid for all positive parameter values.

Idea

Recover (a, b) by solving the limiting system for the first two stationary moments.

Moment-based estimator and consistency

Define

$$\tilde{a}_T = \frac{\sigma^2}{2} \frac{\left(\int_0^T r_t dt\right)^2}{T \int_0^T r_t^2 dt - \left(\int_0^T r_t dt\right)^2},$$
$$\tilde{b}_T = \frac{\sigma^2}{2} \frac{T \int_0^T r_t dt}{T \int_0^T r_t^2 dt - \left(\int_0^T r_t dt\right)^2}.$$

Theorem 5

For all $a, b, \sigma > 0$,

$$(\tilde{a}_T, \tilde{b}_T) \rightarrow (a, b) \quad \text{a.s. as } T \rightarrow \infty.$$

Asymptotic normality

Theorem 6

$$\sqrt{T} \begin{pmatrix} \tilde{a}_T - a \\ \tilde{b}_T - b \end{pmatrix} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma_{\text{mom}}), \quad T \rightarrow \infty,$$

where

$$\Sigma_{\text{mom}} = \begin{pmatrix} \frac{a}{b}(2a + \sigma^2) & 2a + \sigma^2 \\ 2a + \sigma^2 & \frac{2b}{a}(a + \sigma^2) \end{pmatrix}.$$

Takeaway

The moment-based estimator has the usual $\sqrt{T}$ -rate and imposes no additional restriction on the parameters.

MLE vs. moment-based estimator

We compare the asymptotic covariance matrices.

MLE

$$\Sigma_{\text{mle}} = \begin{pmatrix} \frac{a}{b}(2a - \sigma^2) & 2a - \sigma^2 \\ 2a - \sigma^2 & 2b \end{pmatrix}$$

- Requires $2a > \sigma^2$.
- Smaller asymptotic variance.

Moment-based estimator

$$\Sigma_{\text{mom}} = \begin{pmatrix} \frac{a}{b}(2a + \sigma^2) & 2a + \sigma^2 \\ 2a + \sigma^2 & \frac{2b}{a}(a + \sigma^2) \end{pmatrix}$$

- No additional restrictions.
- Simpler statistics.

Conclusion

The MLE is more efficient when it is well defined, while the moment-based estimator is universally applicable.

Discrete moment-based estimator

For the same grid $t_k = k\delta_n$, define

$$\begin{aligned}\tilde{a}_n^d &= \frac{\sigma^2}{2} \frac{\left(\sum_{k=0}^{m_n-1} r_{t_k}\right)^2}{m_n \sum_{k=0}^{m_n-1} r_{t_k}^2 - \left(\sum_{k=0}^{m_n-1} r_{t_k}\right)^2}, \\ \tilde{b}_n^d &= \frac{\sigma^2}{2} \frac{m_n \sum_{k=0}^{m_n-1} r_{t_k}}{m_n \sum_{k=0}^{m_n-1} r_{t_k}^2 - \left(\sum_{k=0}^{m_n-1} r_{t_k}\right)^2}.\end{aligned}$$

Structure

This is the direct discrete counterpart of the continuous moment-based estimator.

Discrete moment-based estimator: asymptotics

Theorem 7

For every $\beta > 0$,

$$(\tilde{a}_n^d, \tilde{b}_n^d) \rightarrow (a, b) \quad \text{a.s. as } n \rightarrow \infty.$$

Theorem 8

If $\beta > 1$, then

$$\sqrt{n} \begin{pmatrix} \tilde{a}_n^d - a \\ \tilde{b}_n^d - b \end{pmatrix} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma_{\text{mom}}),$$

where Σ_{mom} is the same covariance matrix as in the continuous-time result.

Takeaway

Discrete observations preserve the same asymptotic structure as in continuous time.



Extension to the CKLS model

CKLS model and ergodicity

We consider the Chan–Karolyi–Longstaff–Sanders model

$$dr_t = (a - br_t) dt + \sigma r_t^\beta dW_t, \quad r_0 > 0, \quad \frac{1}{2} < \beta < 1.$$

The case $\beta = \frac{1}{2}$ corresponds to the CIR process.

Basic properties

Under $a, b, \sigma > 0$, the equation has a unique positive strong solution. The process is ergodic with stationary density p_∞ , and

$$\pi_q := \int_0^\infty x^q p_\infty(x) dx < \infty \quad \text{for all } q \in \mathbb{R}.$$

The drift parameters (a, b) are estimated under continuous observation; β and σ are treated as known here, but can be identified pathwise from quadratic variation.

MLE for CKLS

The likelihood approach extends directly to the CKLS model. The MLE $(\hat{a}_T, \hat{b}_T)$ is explicit and uses the path functionals

$$\int_0^T r_t^{-2\beta} dt, \quad \int_0^T r_t^{1-2\beta} dt, \quad \int_0^T r_t^{2-2\beta} dt,$$

together with the corresponding Itô integrals with respect to r .

Theorem 9

$$(\hat{a}_T, \hat{b}_T) \rightarrow (a, b) \quad a.s., \quad \sqrt{T} \begin{pmatrix} \hat{a}_T - a \\ \hat{b}_T - b \end{pmatrix} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma_{\text{mle}}),$$

where

$$\Sigma_{\text{mle}} = \sigma^2 \Sigma^{-1}, \quad \Sigma = \begin{pmatrix} \pi_{-2\beta} & -\pi_{1-2\beta} \\ -\pi_{1-2\beta} & \pi_{2-2\beta} \end{pmatrix}.$$

Moment-based estimator for CKLS

For $q \in \mathbb{R}$, let

$$m_{q,T} := \frac{1}{T} \int_0^T r_t^q dt.$$

The invariant moments satisfy

$$\pi_1 = \frac{a}{b}, \quad \pi_{3-2\beta} - \frac{a}{b} \pi_{2-2\beta} = \frac{\sigma^2(1-\beta)a}{b^2}.$$

Replacing invariant moments by their empirical counterparts gives

$$\tilde{a}_T = \frac{\sigma^2(1-\beta)m_{1,T}^2}{m_{3-2\beta,T} - m_{1,T}m_{2-2\beta,T}}, \quad \tilde{b}_T = \frac{\sigma^2(1-\beta)m_{1,T}}{m_{3-2\beta,T} - m_{1,T}m_{2-2\beta,T}}.$$

Theorem 10

$$(\tilde{a}_T, \tilde{b}_T) \rightarrow (a, b) \quad \text{a.s. as } T \rightarrow \infty.$$

Asymptotic normality

Theorem 11

As $T \rightarrow \infty$,

$$\sqrt{T} \begin{pmatrix} \tilde{a}_T - a \\ \tilde{b}_T - b \end{pmatrix} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma_{\text{mom}}),$$

where

$$\Sigma_{\text{mom}} = \frac{b^4}{a^2 \sigma^2 (1 - \beta)^2} A \Gamma A^\top,$$

with

$$A = \begin{pmatrix} \pi_{3-2\beta} & \pi_1 \\ \pi_{2-2\beta} & 1 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \pi_{2\beta} & -\pi_2 \\ -\pi_2 & \pi_{4-2\beta} \end{pmatrix}.$$

Moreover, $\Sigma_{\text{mom}} - \Sigma_{\text{mle}}$ is positive definite; hence the MLE is asymptotically more efficient.

Conclusions

CIR model

- ▶ The MLE is strongly consistent for $2a \geq \sigma^2$ and asymptotically normal for $2a > \sigma^2$.
- ▶ The moment-based estimator is strongly consistent and asymptotically normal for all $a, b, \sigma > 0$ and avoids inverse powers and stochastic integrals.
- ▶ Under suitable high-frequency sampling, the discrete estimators are asymptotically equivalent to their continuous-time counterparts.

CKLS model

- ▶ For $\frac{1}{2} < \beta < 1$, both the MLE and the moment-based estimator are strongly consistent and asymptotically normal.
- ▶ The MLE has the smaller asymptotic covariance matrix, while the moment-based estimator has a simpler structure.