

Milstein-type Schemes for Hyperbolic SPDEs

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What has been done so far?

Stochastic Schrödinger equation

- + exponential Euler
- + additive noise
- = **rate 1**

[Anton, Cohen 2018]



Stochastic Schrödinger equation

- + exponential Euler
- + multiplicative noise
- = **rate 1/2**

[Anton, Cohen 2018]



Stochastic Maxwell's equations

- + exponential Euler
- + multiplicative noise
- = **rate 1/2**

[Cohen, Cui et al. 2020]



Stochastic Schrödinger/Maxwell's

- + implicit E. / Crank–Nicolson
- + multiplicative noise
- = **rate 1/2 up to $\sqrt{\log}$**

[K., Veraar 2024]



Can we reach higher rates for multiplicative noise?

Exponential Euler and Rational Schemes

Stochastic Evolution Equation

X, H Hilbert spaces, $p \in [2, \infty)$.

$$\begin{cases} dU = (-AU + F(U))dt + G(U)dW_t, \\ U(0) = \xi \in L^p(\Omega; X) \end{cases}$$

- $F : X \rightarrow X, G : X \rightarrow \mathcal{L}_2(H, X)$
- W is H -cylindrical Brownian motion
- $-A$ gen. contractive C_0 -semigroup S

Time Discretisation: easy case

$R : [0, \infty) \rightarrow \mathcal{L}(X)$ scheme, $h > 0$ time step,
 $R_h \approx S(h)$ yields $U(0) := \xi$

$$U_j := R_h U_{j-1} + h R_h F(U_{j-1}) + R_h G(U_{j-1}) \Delta W_j$$

- contractive $\|R_h\| \leq 1 \quad \forall h > 0$
- $\Delta W_j := W_{t_j} - W_{t_{j-1}}$
- uniform grid $\{t_j = jh : 0 \leq j \leq M\}$

Our Question:

When do we get convergence of which rate for the
pathwise uniform error $\mathbb{E}(\sup_{j=0, \dots, M} \|U(t_j) - U_j\|_X^p)$?

(!) $\neq$ simpler **pointwise strong error** $\sup_{j=0, \dots, M} \mathbb{E}\|U(t_j) - U_j\|_X^p$

Exponential Euler and Rational Schemes

Optimality of the rate

Rate limited by $h^{1/2}\sqrt{\log(T/k)}$ on $[0, T]$
= Lévy's modulus of continuity of W

$\Rightarrow$ **optimal rate** $\frac{1}{2}$ among schemes using
only first-order Wiener increments ΔW_j
[Müller-Gronbach '02]

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How to overcome the order 1/2 barrier

For strong solution: $h > 0$ time step,

$$U_h = \xi + \int_0^h (-AU_s + F(U_s)) ds + \int_0^h G(U_s) dW_s$$

Approximate temporal integrand to order 0:

$$-AU_s + F(U_s) \approx -AU_0 + F(U_0)$$

Approximate stochastic integrand to order 1/2:

$$\begin{aligned} G(U_s) &\approx G(U_0) + \int_0^s G'(U_r) dU_r \\ &\approx G(U_0) + \int_0^s G'(U_r)(-AU_r + F(U_r)) dr + \int_0^s G'(U_r)G(U_r) dW_r \\ &\approx G(U_0) + \int_0^s G'(U_0)G(U_0) dW_r. \end{aligned}$$

$$\Rightarrow U_h \approx \xi - hAU_0 + hF(U_0) + \int_0^h G(U_0) dW_s + \int_0^h \int_0^s (G'G)(U_0) dW_r dW_s$$

The Milstein scheme

$$\Rightarrow U_h \approx \xi - hAU_0 + hF(U_0) + \int_0^h G(U_0) dW_s + \int_0^h \int_0^s (G'G)(U_0) dW_r dW_s$$

Definition: The (exponential) Milstein scheme [Jentzen, Röckner, '15]

$u = (u_j)_{j=0,\dots,M}$ with $u_0 := \xi$ and

$$u_{j+1} := S(h)u_j + hS(h)F(u_j) + S(h)\left(\int_{t_j}^{t_{j+1}} G(u_j) dW_s + \int_{t_j}^{t_{j+1}} \int_{t_j}^s (G'G)(u_j) dW_r dW_s\right).$$

Definition: The rational Milstein scheme [Da Prato, Jentzen, Röckner '12]

$u = (u_j)_{j=0,\dots,M}$ with $u_0 := \xi$ and for $(R_h)_{h>0}$ with $R_h \approx S(h)$

$$u_{j+1} := R_h u_j + hR_h F(u_j) + R_h \left(\int_{t_j}^{t_{j+1}} G(u_j) dW_s + \int_{t_j}^{t_{j+1}} \int_{t_j}^s (G'G)(u_j) dW_r dW_s \right)$$

$$\Rightarrow u_j = R_h^j \xi + h \sum_{i=0}^{j-1} R_h^{j-i} F(u_i) + \sum_{i=0}^{j-1} R_h^{j-i} \left(\int_{t_i}^{t_{i+1}} G(u_i) dW_s + \int_{t_i}^{t_{i+1}} \int_{t_i}^s (G'G)(u_i) dW_r dW_s \right).$$

Literature Overview

What is known for parabolic SPDEs?

- convergence of the **root-mean-square error at rate 1** [Jentzen, Röckner, '15]
- parabolic SPDEs driven by non-continuous martingale noise [Barth, Lang '12]
- L^p - **and almost sure convergence** for advection-diffusion equations [Barth, Lang '13]
- **non-commutative noise** in [von Hallern, Rößler '20]
- **space-time discretisation** by a Milstein–Galerkin scheme [Kruse '14]
- software package for numerical approximation of Lévy areas [Kastner, Rößler '23]

Our contributions:

- first optimal convergence rates for **hyperbolic SPDEs** (partially answers an open problem in [J., R. '15]),
- **rational Milstein schemes** with $R_h \neq S(h)$ (answers open problem in [J., R. '15])
- maximal estimates in p -th moment, $p \in [2, \infty) \rightsquigarrow$ **pathwise uniform** convergence rates

Assumptions

Let $Y \hookrightarrow D(A^\alpha) \hookrightarrow X$ Hilbert space for some $\alpha \in (\frac{1}{2}, 1]$. Abbreviate $\|\cdot\|_{\mathcal{L}_2(H,X)}$ by $|||\cdot|||_X$.

- **LipX (Global Lipschitz continuity on X):** for all $x, y \in X$

$$\|F(x) - F(y)\|_X + |||G(x) - G(y)|||_X \lesssim \|x - y\|_X$$

- **LinGroY (Linear growth on Y):** for all $x \in Y$

$$\|F(x)\|_Y + |||G(x)|||_Y \lesssim 1 + \|x\|_Y$$

- **GâtX (Gâteaux differentiability):** $F: X \rightarrow X$ and $G: X \rightarrow \mathcal{L}_2(H, X)$ are Gâteaux diffble
- **HölderYX(α) (($2\alpha - 1$)-Hölder continuity of derivatives $F'|_Y, G'|_Y$):** Gâteaux derivatives $F'|_Y: Y \rightarrow \mathcal{L}(Y, X)$ and $G'|_Y: Y \rightarrow \mathcal{L}(Y, \mathcal{L}_2(H, X))$ are $(2\alpha - 1)$ -Hölder: for all $x, y, z \in Y$

$$\|(F'(x) - F'(y))[z]\|_X + |||(G'(x) - G'(y))[z]|||_X \lesssim \|x - y\|_Y^{2\alpha-1} \|z\|_Y$$

- **MilCor (Lipschitz continuity and linear growth of Milstein correction):**

$G' \circ G: X \rightarrow \mathcal{L}_2^{(2)}(H, X)$ is Lipschitz continuous and

$G' \circ G: Y \rightarrow \mathcal{L}_2^{(2)}(H, Y)$ is of linear growth.

The main theorem: Exponential Milstein scheme

Theorem [Kastner, K., '25, arXiv2512.19647]

- X, H Hilbert spaces, $p \in [2, \infty)$,
- $-A$ generates a contractive C_0 -semigroup $(S(t))_{t \geq 0}$ on X and Y ,
- $Y \hookrightarrow D(A^\alpha) \hookrightarrow X$ Hilbert space for some $\alpha \in (\frac{1}{2}, 1]$, $\xi \in L^{2\alpha p}(\Omega; Y)$,
- $(u_j)_j$ exponential Milstein scheme,
- Assumptions (LipX), (LinGroY), (GâtX), (HölDerYX(α)), (MilCor).

Then

$$\left(\mathbb{E} \left(\sup_{j=0, \dots, M} \|U(t_j) - u_j\|_X^p \right) \right)^{1/p} \leq C h^\alpha.$$

$\Rightarrow$ convergence at **optimal rate 1**

- also works for $F = F(\omega, t, x)$, $G = G(\omega, t, x)$ & in 2-smooth Banach spaces
- additive noise: Milstein = exponential Euler

The main theorem: Rational Milstein schemes

Theorem [Kastner, K., '25, arXiv2512.19647]

- X, H Hilbert spaces, $p \in [2, \infty)$,
- $-A$ generates a contractive C_0 -semigroup $(S(t))_{t \geq 0}$ on X and Y ,
- $Y \hookrightarrow D(A^\alpha) \hookrightarrow X$ Hilbert space for some $\alpha \in (\frac{1}{2}, 1]$, $\xi \in L^{2\alpha p}(\Omega; Y)$,
- $(R_h)_{h>0}$ **contractive scheme** on X, Y with $\|[S(t_j) - R_h^j]x\|_X \lesssim h^\alpha \|x\|_Y$,
- $(u_j)_j$ **rational** Milstein scheme,
- Assumptions (LipX), (LinGroY), (GâtX), (HölderYX(α)), (MilCor).

Then

$$\left(\mathbb{E} \left(\sup_{j=0, \dots, M} \|U(t_j) - u_j\|_X^p \right) \right)^{1/p} \leq C \sqrt{\log \left(\frac{T}{h} \right)} \cdot h^\alpha.$$

$\Rightarrow$ convergence at **optimal rate 1 up to log factor**

- also works for $F = F(\omega, t, x)$, $G = G(\omega, t, x)$ & in 2-smooth Banach spaces
- additive noise: Milstein = **corresponding rational scheme**

What does this imply?

Different admissible choices of $(R_h)_{h>0}$:

- exponential Euler (EXP): $R_h = S(h)$
- implicit Euler (IE): $R_h = (1 + hA)^{-1}$, rate α on $D(A^{2\alpha})$
- Crank–Nicolson (CN): $R_h = (2 - hA)(2 + hA)^{-1}$, rate α on $D(A^{3\alpha/2})$

Rates on $Y = D(A^\beta)$, $\beta > 0$:

	EXP	IE	CN	EXP-Mil	IE-Mil	CN-Mil
multiplicative noise	β	$\beta/2$	$(2\beta)/3$	β	$\beta/2$	$(2\beta)/3$
optimal rate	$1/2$	$1/2$	$1/2$	1	1	1
Y for optimal rate	$D(A^{1/2})$	$D(A)$	$D(A^{3/4})$	$D(A)$	$D(A^2)$	$D(A^{3/2})$

Theorem (Kastner, K. '25): Continuous extension on $[0, T]$

For **continuous extension** $\bar{u}$ of rational Milstein scheme $(u_j)_j$:

Rate up to $h\sqrt{\log(T/h)}$ on $[0, T]$

piecewise constant/affine linear interpolations: rate limited to $h^{1/2}\sqrt{\log(T/h)}$

How the Milstein term helps in the proof

Abbreviate $V_s^i := S(s - t_i)U_{t_i}$ for $s \in [t_i, t_{i+1})$. Taylor expansion:

$$G(U_s) = G(V_s^i) + G'(V_s^i)[U_s - V_s^i] + \int_0^1 (G'(V_s^i + \zeta(U_s - V_s^i)) - G'(V_s^i))[U_s - V_s^i] d\zeta$$

One of the error terms:

$$\begin{aligned} E_G &:= \left(\sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left\| G(U_s) - G(V_s^i) - \int_{t_i}^s (G'G)(V_s^i) dW_r \right\|_{p,X}^2 ds \right)^{1/2} \\ &\lesssim \left(\sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left\| G'(V_s^i) \left[U_s - V_s^i - \int_{t_i}^s G(V_s^i) dW_r \right] \right\|_{p,X}^2 ds \right)^{1/2} \\ &\quad + \left(\sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left\| \int_0^1 (G'(V_s^i + \zeta(U_s - V_s^i)) - G'(V_s^i))[U_s - V_s^i] d\zeta \right\|_{p,X}^2 ds \right)^{1/2} \\ &=: E_{G,1} + E_{G,2}. \end{aligned}$$

How the Milstein term helps in the proof

1st term: $(\hat{G}t) \Rightarrow$ uniform bddness of G'

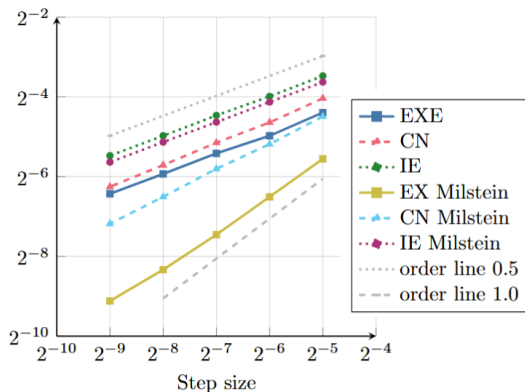
& higher-order decay of difference with Milstein correction in X

$$\begin{aligned} E_{G,1} &= \left(\sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left\| \left\| G'(V_s^i) \left[U_s - V_s^i - \int_{t_i}^s G(V_s^i) dW_r \right] \right\|_{p,X}^2 ds \right)^{1/2} \\ &\lesssim \left(\sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left\| \left\| G'(V_s^i) \right\|_{L(X, L_2(H, X))} \left\| U_s - V_s^i - \int_{t_i}^s G(V_s^i) dW_r \right\|_X \right\|_p^2 ds \right)^{1/2} \\ &\lesssim \left(\sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left((s - t_i)^{\frac{1}{2} + \frac{1}{2}} + (s - t_i)^{\alpha + \frac{1}{2}} \right)^2 ds \right)^{1/2} \lesssim \sqrt{T} (h + h^{\alpha + \frac{1}{2}}). \end{aligned}$$

2nd term: $(2\alpha - 1)$ -Hölder continuity of $G': Y \rightarrow \mathcal{L}(Y, \mathcal{L}_2(H, X))$ and lower-order decay in Y

$$\begin{aligned} E_{G,2} &= \left(\sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left\| \int_0^1 (G'(V_s^i + \zeta(U_s - V_s^i)) - G'(V_s^i)) [U_s - V_s^i] d\zeta \right\|_{p,X}^2 ds \right)^{1/2} \\ &\lesssim \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \int_0^1 \zeta^{2\alpha-1} \|U_s - V_s^i\|_{2\alpha p, Y}^{2\alpha} d\zeta ds \lesssim \sqrt{T} h^\alpha. \end{aligned}$$

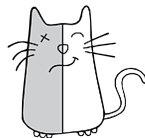
Numerical simulation: Schrödinger with a nonlocal nonlinearity



Example: linear multiplicative noise, Q has eigenvalues $\lambda_r = (1 + |r|^{5.1})^{-1}$ to $e_r = \exp(ir\cdot)$, $r \in \mathbb{Z}$, η smooth bump, nonlocal drift $F(u) = -i\eta * \frac{u}{1+|u|^2}$

expected rates: numerical rates:

- | | |
|-------------|--------------|
| • 1 (EXPM) | • 0.89 (EXP) |
| • 2/3 (CNM) | • 0.67 (CN) |
| • 1/2 (IE) | • 0.50 (IE) |



What's next?

Want to know more?

arXiv 2512.19647
accepted at ESAIM:M2AN



Next steps

- Schrödinger with Nemytskii nonlinearity: $Y = H^2$ needs linear $\leadsto$ polynomial growth
- globally $\leadsto$ locally Lipschitz tamed exponential Euler
- semi-linear
 - $\leadsto$ non-autonomous: $A = A(t)$
 - $\leadsto$ quasi-linear: $A = A(U)$

Rational schemes

IMA J.Numer.Anal., '24
with Mark Veraar



Hvala vam na pažnji!