

Splitting methods for stochastic Hodgkin-Huxley type systems

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joint work with Pierre Étoré (Grenoble Alpes) and Anna Melnykova (Avignon)



Introduction & Motivation

Deterministic Hodgkin-Huxley system

4-dim. ODE, describing how action potentials are generated in a neuron:

$$dV_t = \frac{1}{C} (I - g_K n_t^4 (V_t - E_K) - g_{Na} m_t^3 h_t (V_t - E_{Na}) - g_L (V_t - E_L)) dt,$$

$$dn_t = \alpha_n(V_t)(1 - n_t) - \beta_n(V_t)n_t dt,$$

$$dm_t = \alpha_m(V_t)(1 - m_t) - \beta_m(V_t)m_t dt,$$

$$dh_t = \alpha_h(V_t)(1 - h_t) - \beta_h(V_t)h_t dt.$$

- ▶ Membrane potential: $V_t \in \mathbb{R}$
- ▶ Auxiliary components: $n_t, m_t, h_t \in [0, 1]$

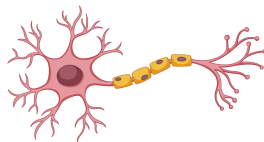


Figure: Designed by brgfx / Freepik

Hodgkin & Huxley (J. Physiol., 1952): *A quantitative description of membrane current and its application to conduction and excitation in nerve.*

Deterministic Hodgkin-Huxley system: Splitting methods

Chen, Raman & Stern (SIAM J. Sci. Comput., 2020): *Structure-preserving numerical integrators for Hodgkin-Huxley-Type systems.*

Deterministic Hodgkin-Huxley system: Splitting methods

Conditionally linear system:

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Stochastic Hodgkin-Huxley type systems

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System **reformulation**:

$$\begin{aligned}dV_t &= \frac{1}{C} (I - g_K n_t^4 (V_t - E_K) - g_{Na} m_t^3 h_t (V_t - E_{Na}) - g_L (V_t - E_L)) dt, \\dn_t &= \alpha_n(V_t)(1 - n_t) - \beta_n(V_t)n_t dt, \\dm_t &= \alpha_m(V_t)(1 - m_t) - \beta_m(V_t)m_t dt, \\dh_t &= \alpha_h(V_t)(1 - h_t) - \beta_h(V_t)h_t dt.\end{aligned}$$

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Define:

$$\begin{aligned}U_t &:= (n_t, m_t, h_t)^\top \in [0, 1]^3, \\a(U_t) &:= \frac{1}{C} (-g_K n_t^4 - g_{Na} m_t^3 h_t - g_L) \in (-\infty, 0), \\b(U_t) &:= \frac{1}{C} (I + g_K E_K n_t^4 + g_{Na} E_{Na} m_t^3 h_t + g_L E_L) \in \mathbb{R}.\end{aligned}$$

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Then:

$$dV_t = (a(U_t)V_t + b(U_t))dt$$

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$$\begin{aligned}dV_t &= (a(U_t)V_t + b(U_t))dt + \Sigma(U_t)dW_t, \\dU_t &= (-\text{diag}(\alpha(V_t) + \beta(V_t))U_t + \alpha(V_t))dt.\end{aligned}$$

Stochastic Hodgkin-Huxley type systems

$(d + 1)$ -dimensional SDE:

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- ▶ $a : \mathbb{R}^d \rightarrow (-\infty, 0)$, $b : \mathbb{R}^d \rightarrow \mathbb{R}$, $\Sigma : \mathbb{R}^d \rightarrow \mathbb{R}$, and $\alpha, \beta : \mathbb{R} \rightarrow (0, \infty)^d$ are C^1 functions.
- ▶ Locally Lipschitz coefficients.

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- ▶ Locally Lipschitz coefficients.
- ▶ Existence of a unique strong solution $X = (V, U^\top)^\top$ on $[0, T]$ with

$$U_t \in [0, 1]^d, \quad 0 \leq t \leq T, \quad \text{a.s.},$$

and

$$\mathbb{E} \left[\max_{0 \leq t \leq T} |V_t|^{2p} \right] \leq C (1 + \mathbb{E} [|V_0|^{2p}]), \quad p \geq 1.$$

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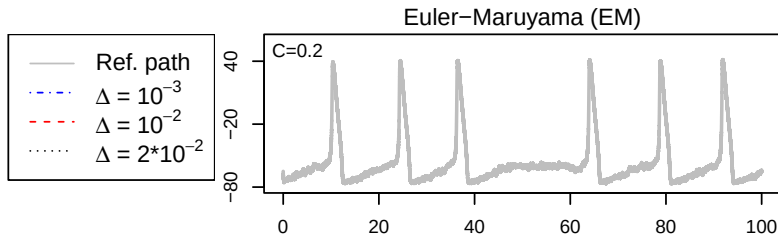
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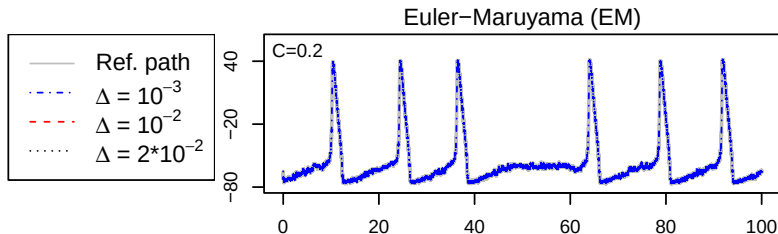
Höpfner, Löcherbach, and Thieullen (2012): *Transition densities for stochastic Hodgkin-Huxley models*.

Stochastic Hodgkin-Huxley system: Euler-Maruyama type methods

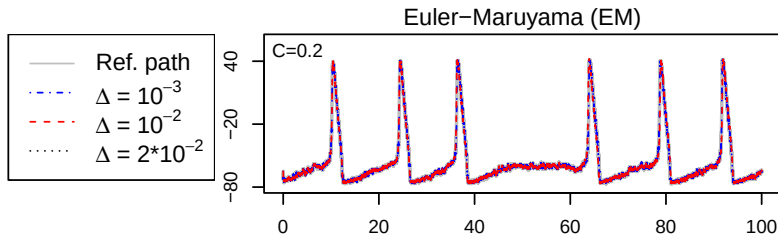
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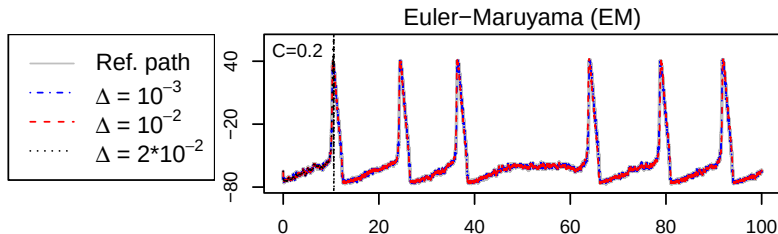
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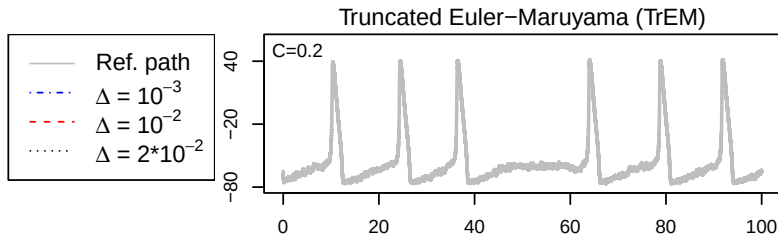
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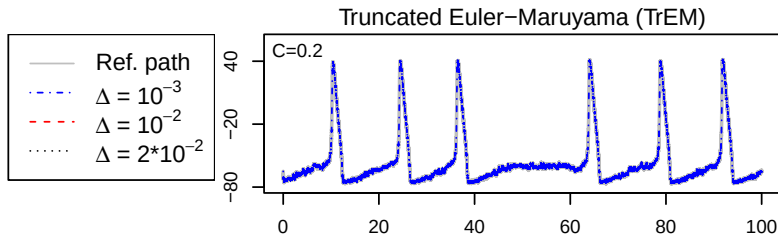
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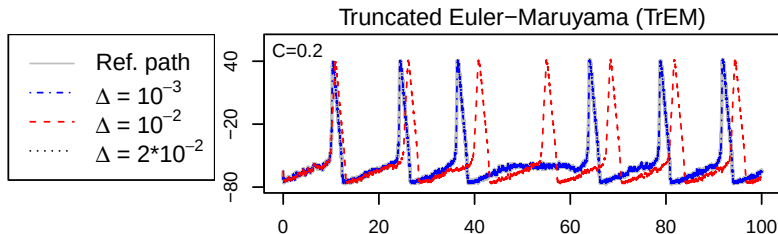
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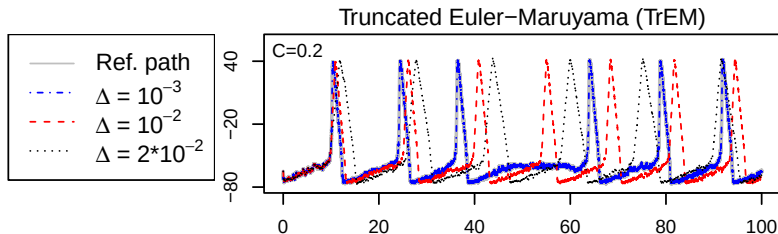
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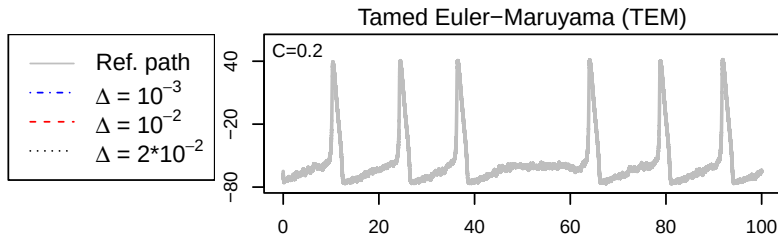
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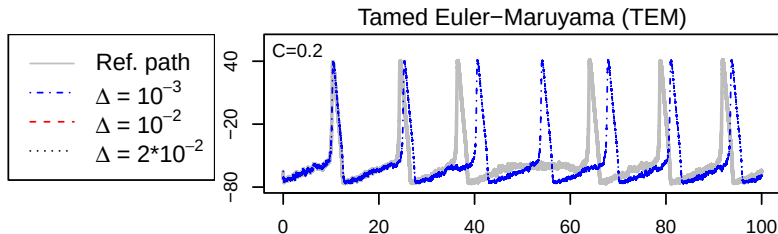
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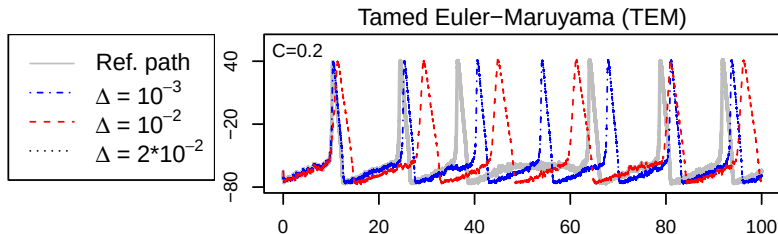
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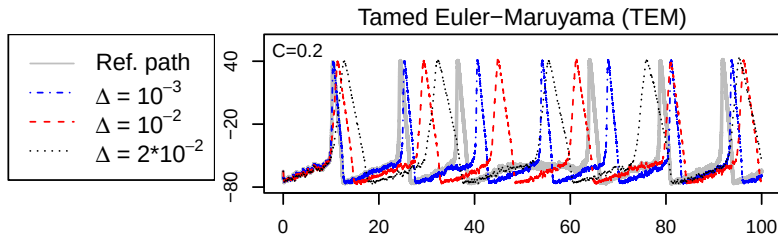
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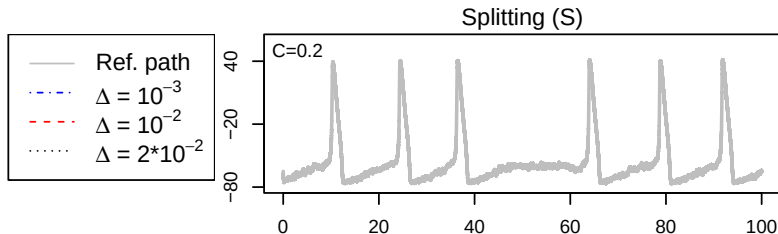
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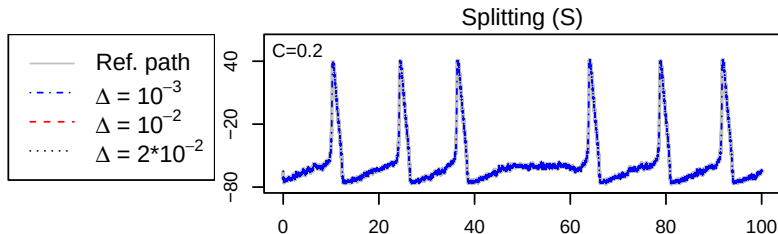
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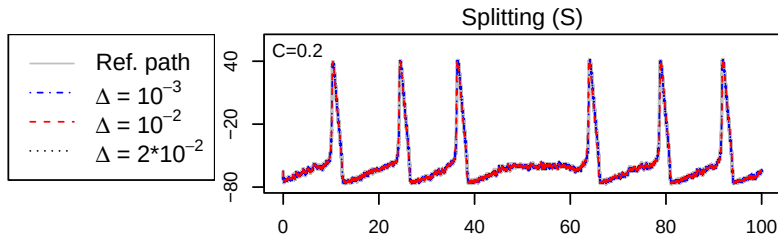
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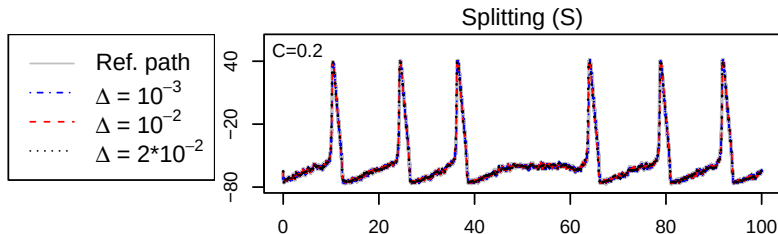
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Stochastic Hodgkin-Huxley system: Splitting methods



Outline

- ▶ Part I: Construction of splitting methods,
- ▶ Part II: Structure-preservation,
- ▶ Part III: Mean-square convergence.

Part I

Construction of splitting methods

inspired by Chen, Raman & Stern (SIAM J. Sci. Comput., 2020):
Structure-preserving numerical integrators for Hodgkin–Huxley–Type systems.

Step 1: choice of subsystems

Conditionally linear system:

$$\begin{aligned}dV_t &= (a(U_t) V_t + b(U_t)) dt + \Sigma(U_t) dW_t, \\dU_t &= (-\text{diag}(\alpha(V_t) + \beta(V_t)) U_t + \alpha(V_t)) dt.\end{aligned}$$

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Subsystems:

- $d \begin{pmatrix} V_t^{[1]} \\ U_t^{[1]} \end{pmatrix} = \begin{pmatrix} a(U_t^{[1]}) V_t^{[1]} + b(U_t^{[1]}) \\ 0_d \end{pmatrix} dt + \begin{pmatrix} \Sigma(U_t^{[1]}) \\ 0_d \end{pmatrix} dW_t$
- $d \begin{pmatrix} V_t^{[2]} \\ U_t^{[2]} \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathbf{diag}(\alpha(V_t^{[2]}) + \beta(V_t^{[2]})) U_t^{[2]} + \alpha(V_t^{[2]}) \end{pmatrix} dt$

Step 2: solutions of subsystems

- **First** subsystem:

$$d \begin{pmatrix} V_t^{[1]} \\ U_t^{[1]} \end{pmatrix} = \begin{pmatrix} a(U_t^{[1]})V_t^{[1]} + b(U_t^{[1]}) \\ 0_d \end{pmatrix} dt + \begin{pmatrix} \Sigma(U_t^{[1]}) \\ 0_d \end{pmatrix} dW_t.$$

Explicit solution:

$$\phi_t^{[1;\text{Stoch}]}(x) = \begin{pmatrix} \psi_{\text{Stoch}}(t; v, u) \\ u \end{pmatrix},$$

where

$$\psi_{\text{Stoch}}(t; v, u) := e^{a(u)t}v + \underbrace{b(u) \int_0^t e^{a(u)(t-s)} ds + \Sigma(u) \int_0^t e^{a(u)(t-s)} dW_s}_{= \frac{b(u)}{a(u)}(e^{a(u)t} - 1)}.$$

Step 2: solutions of subsystems

- **Second** subsystem:

$$d \begin{pmatrix} V_t^{[2]} \\ U_t^{[2]} \end{pmatrix} = \begin{pmatrix} 0 \\ -\text{diag} \left(\alpha(V_t^{[2]}) + \beta(V_t^{[2]}) \right) U_t^{[2]} + \alpha(V_t^{[2]}) \end{pmatrix} dt.$$

Explicit solution:

$$\phi_t^{[2;\text{Det}]}(x) = \begin{pmatrix} v \\ \psi_{\text{Det}}(t; v, u) \end{pmatrix},$$

where

$$\begin{aligned} \psi_{\text{Det}}(t; v, u) &:= e^{-\text{diag}(\alpha(v)+\beta(v))t} u + \underbrace{\int_0^t e^{-\text{diag}(\alpha(v)+\beta(v))(t-s)} \alpha(v) ds}_{= \left(\mathbb{I}_d - e^{-\text{diag}(\alpha(v)+\beta(v))t} \right) U^\infty(v)}, \end{aligned}$$

and

$$U^\infty(v) := \left(U^{1,\infty}(v), \dots, U^{d,\infty}(v) \right)^\top, \quad U^{l,\infty}(v) = \frac{\alpha_l(v)}{\alpha_l(v) + \beta_l(v)}, \quad l \in \{1, \dots, d\}.$$

Step 3: iterative composition of solutions

- Equidistant time discretisation: $0 = t_0 < t_1 < \dots < t_n = T$, $\Delta = t_i - t_{i-1}$
- **Splitting method:**

$$\tilde{X}_{t_i} = \left(\phi_{\Delta}^{[2;\text{Det}]} \circ \phi_{\Delta}^{[1;\text{Stoch}]} \right) (\tilde{X}_{t_{i-1}}), \quad \tilde{X}_{t_0} := X_0$$

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Specifically, we have:

$$\begin{aligned}\tilde{V}_{t_i} &= \psi_{\text{Stoch}}(\Delta; \tilde{V}_{t_{i-1}}, \tilde{U}_{t_{i-1}}), \\ \tilde{U}_{t_i} &= \psi_{\text{Det}}(\Delta; \tilde{V}_{t_i}, \tilde{U}_{t_{i-1}}).\end{aligned}$$

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Remember:

$$\begin{aligned}\psi_{\text{Stoch}}(t; v, u) &= e^{a(u)t} v + \frac{b(u)}{a(u)} \left(e^{a(u)t} - 1 \right) + \Sigma(u) \int_0^t e^{a(u)(t-s)} dW_s, \\ \psi_{\text{Det}}(t; v, u) &= e^{-\text{diag}(\alpha(v)+\beta(v))t} u + \left(\mathbb{I}_d - e^{-\text{diag}(\alpha(v)+\beta(v))t} \right) U^{\infty}(v).\end{aligned}$$

Part II

Structure-preservation

Structure-preservation

Splitting method:

$$\tilde{V}_{t_i} = \psi_{\text{Stoch}}(\Delta; \tilde{V}_{t_{i-1}}, \tilde{U}_{t_{i-1}}),$$

$$\tilde{U}_{t_i} = \psi_{\text{Det}}(\Delta; \tilde{V}_{t_i}, \tilde{U}_{t_{i-1}}),$$

where

$$\psi_{\text{Stoch}}(t; v, u) = e^{a(u)t} v + \frac{b(u)}{a(u)} \left(e^{a(u)t} - 1 \right) + \Sigma(u) \int_0^t e^{a(u)(t-s)} dW_s,$$

$$\psi_{\text{Det}}(t; v, u) = e^{-\text{diag}(\alpha(v)+\beta(v))t} u + \left(\mathbb{I}_d - e^{-\text{diag}(\alpha(v)+\beta(v))t} \right) U^\infty(v).$$

Structure-preservation

Splitting method:

$$\begin{aligned}\tilde{V}_{t_i} &= \psi_{\text{Stoch}}(\Delta; \tilde{V}_{t_{i-1}}, \tilde{U}_{t_{i-1}}), \\ \tilde{U}_{t_i} &= \psi_{\text{Det}}(\Delta; \tilde{V}_{t_i}, \tilde{U}_{t_{i-1}}),\end{aligned}$$

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$$\begin{aligned}\psi_{\text{Stoch}}(t; v, u) &= e^{a(u)t} v + \frac{b(u)}{a(u)} \left(e^{a(u)t} - 1 \right) + \Sigma(u) \int_0^t e^{a(u)(t-s)} dW_s, \\ \psi_{\text{Det}}(t; v, u) &= e^{-\text{diag}(\alpha(v)+\beta(v))t} u + \left(\mathbb{I}_d - e^{-\text{diag}(\alpha(v)+\beta(v))t} \right) U^\infty(v).\end{aligned}$$

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*Mattingly, Stuart & Higham (Stoch. Process. Their Appl., 2002): *Ergodicity for SDEs and approximations: Locally Lipschitz vector fields and degenerate noise.*

Part III

Mean-square convergence

Can we use the fundamental mean-square convergence theorem?

Tretyakov & Zhang (SIAM J. Numer. Anal., 2013): *A fundamental mean-square convergence theorem for SDEs with locally Lipschitz coefficients and its applications.*

Can we use the fundamental mean-square convergence theorem?

Assume:

- Both X and $\tilde{X}$ have bounded moments:

$$\mathbb{E} \left[\max_{0 \leq t \leq T} |X_t|^{2p} \right] \leq C (1 + \mathbb{E} [|X_0|^{2p}]) , \quad \mathbb{E} \left[\max_{0 \leq t \leq T} |\tilde{X}_t|^{2p} \right] \leq C (1 + \mathbb{E} [|X_0|^{2p}]) .$$

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- The one-step approximation is consistent:

$$\begin{aligned} \left| \mathbb{E} \left[X_{t_i}^{t_{i-1}, x} - \tilde{X}_{t_i}^{t_{i-1}, x} \right] \right| &\leq C (1 + |x|^{2\alpha})^{1/2} \Delta^{q_1}, \\ \left(\mathbb{E} \left[\left| X_{t_i}^{t_{i-1}, x} - \tilde{X}_{t_i}^{t_{i-1}, x} \right|^{(2p)} \right] \right)^{1/(2p)} &\leq C (1 + |x|^{2\alpha p})^{1/(2p)} \Delta^{q_2}, \end{aligned}$$

with $q_2 \geq 1/2$, and $q_1 \geq q_2 + 1/2$.

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What to do in the locally Lipschitz case?

Higham, Mao & Stuart (SIAM J. Numer. Anal., 2002): *Strong convergence of Euler-type methods for nonlinear stochastic differential equations.*

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Idea of proof:

- ▶ Define the stopping times:

$$\rho_R^X := \inf\{0 \leq t \leq T : |X_t| \geq R\}, \quad \rho_R^{\bar{X}} := \inf\{0 \leq t \leq T : |\bar{X}_t| \geq R\}, \quad \rho_R := \rho_R^{\bar{X}} \wedge \rho_R^X.$$

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Assume in addition: • Both X and $\bar{X}$ have bounded moments.

Then:

$$\sup_{i=0, \dots, n} \mathbb{E} \left[|X_{t_i} - \bar{X}_{t_i}|^2 \right] \xrightarrow[\Delta \downarrow 0]{} 0.$$

Application: splitting methods for stochastic Hodgkin-Huxley type systems

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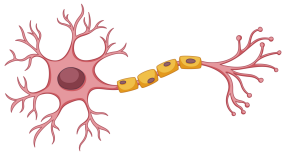


Figure: Designed by brgfx / Freepik

Thank you!



Figure: Scan for arxiv preprint