



Analytical approximations of solutions of SDEs by applying Taylor series

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NOTATIONS:

- $|\cdot|$ is the Frobenius norm of matrices (Euclidean for vectors)
- $\|\cdot\|_m$ is the standard norm of m –linear operators
- $\langle \cdot, \cdot \rangle$ is the standard Euclidean scalar product of vectors
- $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$
- $V = \mathcal{C}([-\tau, 0]; \mathbb{R}^d)$ is a class of continuous functions
- $\mathbb{V} = (V, \|\cdot\|)$ is a Banach space with the supremum norm
- $\mathcal{L}(X, Y)$ is the space of all bounded linear operators
- $\mathcal{M}_i(X^i; Y)$ is the space of all i –linear operators from X^i to Y
- $(\Omega, \mathcal{F}, P)$ is a complete, filtered probability space
- $\{\mathcal{F}_t\}_{t \geq 0}$ is the filtration on the probability space which satisfies the usual conditions (it is non-decreasing, continuous from the right and $\mathcal{F}_0$ contains all P –null sets)
- w is a standard d_1 –dimensional Wiener process, $\mathcal{F}_t$ –adapted

FIELD OF INVESTIGATION:

Strong solutions - solutions on a **fixed** probability space

STOCHASTIC DIFFERENTIAL EQUATION (DIFFERENTIAL FORM)

$$dx(t) = a(t, x(t))dt + b(t, x(t))dw(t), \quad t \in [t_0, T], \quad x(t_0) = x_0. \quad (1)$$

- $0 \leq t_0 < T$ are fixed real numbers
- $a : [t_0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the drift coefficient
- $b : [t_0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d_1}$ is the diffusion coefficient
- a and b are Borel measurable
- x_0 is $\mathcal{F}_{t_0}$ –measurable, $\mathbb{R}^d$ –valued random variable
- $x(t)$ is an $\mathbb{R}^d$ –valued random variable for any $t \in [t_0, T]$

Stochastic differential equations (SDEs) cannot be solved in general, but, they can be solved under some conditions for its coefficients.

THEOREM (SUFFICIENT CONDITIONS FOR EXISTENCE AND UNIQUENESS OF SOLUTION)

If the drift and diffusion coefficients of Eq. (1) satisfy the local Lipschitz condition and the linear growth condition, then there exists a unique solution $x = \{x(t), t \in [t_0, T]\}$ to Eq. (1) with initial condition $x(t_0) = x_0$.

THEOREM (SOME SUFFICIENT CONDITIONS FOR EXISTENCE AND UNIQUENESS OF SOLUTION)

If the drift and diffusion coefficients satisfy both local Lipschitz and monotone condition, then there exists a unique solution to equation (1).

LEMMA

If the drift coefficient a satisfies one-sided Lipschitz condition and the diffusion coefficient b satisfies global Lipschitz condition, then the monotone condition holds for coefficients of the equation (1).

(Higham, Mao, Stuart (2002)):

If the drift coefficient satisfies the one-sided Lipschitz condition and the diffusion coefficient satisfies the global Lipschitz condition, then the stochastic differential equation (1) has a bounded p -th moment for $p > 2$, i.e., for every $p > 2$, there is $C = C(p, T) > 0$, such that

$$E \sup_{t \in [0, T]} |x(t)|^p \leq C(1 + E|x_0|^p).$$

Consider SDE (1) of the Itô type in integral form:

STOCHASTIC DIFFERENTIAL EQUATION (INTEGRAL FORM)

$$x(t) = x_0 + \int_{t_0}^t a(u, x(u)) du + \int_{t_0}^t b(u, x(u)) dw(u), \quad t \in [t_0, T] \quad (2)$$

- Define the arbitrary partition of the time interval $[t_0, T]$:

$$t_0 < t_1 < \dots < t_n = T. \quad (3)$$

- n is large enough such that

$$\delta_n = \max\{t_{i+1} - t_i \mid i \in \{0, \dots, n-1\}\} < 1.$$

GOAL:

Approximation of the solution $x = \{x(t), t \in [t_0, T]\}$ of Eq. (2) with initial condition x_0 on partition (3) via processes noted as $x^n = \{x^n(t), t \in [t_0, T]\}$, which are obtained by successive connecting of solutions of the Eqs.

$$\begin{aligned}
 x^n(t) = & x^n(t_k) + \int_{t_k}^t \sum_{i=0}^{m_1} \frac{a_x^{(i)}(s, x^n(t_k))(x^n(s) - x^n(t_k))^i}{i!} ds \\
 & + \int_{t_k}^t \sum_{i=0}^{m_2} \frac{b_x^{(i)}(s, x^n(t_k))(x^n(s) - x^n(t_k))^i}{i!} dw(s), \\
 & t \in [t_k, t_{k+1}],
 \end{aligned} \tag{4}$$

with the initial data $x^n(t_0) = x_0$ for $k = 0$ and $x^n(t_k)$ for $k \in \{1, \dots, n-1\}$, in division points of the partition (3).

ASSUMPTIONS:

$\mathcal{A}_1$: There exist Taylor expansions of the functionals a and b in the first argument up to the m_1 -th and m_2 -th derivatives, respectively.

$\mathcal{A}_2$: Functions $a_x^{(m_1+1)}$ and $b_x^{(m_2+1)}$ are uniformly bounded, that is, there exist positive constants L_1 and L_2 such that

$$\sup_{(t,x) \in [t_0, T] \times \mathbb{R}^d} \|a_x^{(m_1+1)}(t, x)\|_{m_1+1} \leq L_1, \quad \sup_{(t,x) \in [t_0, T] \times \mathbb{R}^d} \|b_x^{(m_2+1)}(t, x)\|_{m_2+1} \leq L_2.$$

$\mathcal{A}_3$: Functions a and b satisfy the following polynomial condition:
 $(\exists D > 0)(\exists q \in \mathbb{N}_0) (\forall x, y \in \mathbb{R}^d)(\forall t \in [t_0, T]),$

$$|a(t, x) - a(t, y)|^2 \vee |b(t, x) - b(t, y)|^2 \leq D(1 + |x|^q + |y|^q)|x - y|^2.$$

$\mathcal{A}_4$: Functions $a(\cdot, 0)$ and $b(\cdot, 0)$ are bounded on $[t_0, T]$, that is, there exist positive constants C_a and C_b such that $|a(t, 0)| \leq C_a$ and $|b(t, 0)| \leq C_b$, for $t \in [t_0, T]$.

$\mathcal{A}_5$: There exist unique solutions x and x^n of Eqs. (2) and (4), respectively, for appropriate initial conditions such that for some $p > 2$ and $Q > 0$ independent of n , for $M = m_1 \vee m_2$,

$$E \sup_{t \in [t_0, T]} |x(t)|^{p(1+(M \vee q))} < \infty,$$

$$E \sup_{t \in [t_0, T]} |x^n(t)|^{p(1+(M \vee q))^2} \leq Q < \infty.$$

NOTE:

- Some well-known inequalities are used, such as the Hölder and the Burkholder-Davis-Gundy inequality, and some special ones, such as the integral Bihary type inequality (Oguntuase 2000).
- One elementary inequality (corollary of the Hölder inequality) is specifically used in the sequel: for every nonnegative real number r and for all vectors a_i , $i \in \{1, \dots, n\}$, $n \in \mathbb{N}$, from the normed space $(W, \|\cdot\|_W)$,

$$\left\| \sum_{i=1}^n a_i \right\|_W^r \leq (n^{r-1} \vee 1) \sum_{i=1}^n \|a_i\|_W^r. \quad (5)$$

Lemma

Let $\{x^n(t), t \in [t_k, t_{k+1}]\}$, $k \in \{0, \dots, n-1\}$ represent the solutions of Eqs. (4) with the appropriate initial conditions and let us assume that the conditions $\mathcal{A}_1 - \mathcal{A}_5$ are satisfied. Then, for every $0 < r \leq p(1 + (M \vee q))$,

$$E|x^n(t) - x^n(t_k)|^r \leq C'\delta_n^{r/2}, \quad t \in [t_k, t_{k+1}], \quad k \in \{0, \dots, n-1\},$$

where C' is a constant independent of both n and δ_n .

Theorem

Let $x = \{x(t), t \in [t_0, T]\}$ and $\{x^n(t), t \in [t_k, t_{k+1}]\}$, $k \in \{0, \dots, n-1\}$ represent the solutions of Eqs. (2) and (4), respectively, with the appropriate initial conditions. Let us assume that the conditions $\mathcal{A}_1 - \mathcal{A}_5$ are satisfied. Then, for every $p > 0$,

$$E \sup_{t \in [t_0, T]} |x(t) - x^n(t)|^p \leq K \delta_n^{(m+1)p/2},$$

where $m = m_1 \wedge m_2$ and K is a constant independent of n and δ_n .

Theorem

Let the assumptions of Theorem 2 be satisfied and let there exist a decreasing sequence $(\lambda_n)_{n \in \mathbb{N}}$ of positive real numbers such that $\lambda_n \rightarrow 0$ when $n \rightarrow \infty$ and $\sum \delta_n \lambda_n^{-2} < \infty$. Then, the sequence $(x^n)_{n \in \mathbb{N}}$ of the solutions of Eqs. (4) converges almost surely to the solution x of Eq. (2).

EXAMPLE

Autonomous stochastic differential equation:

$$dx(t) = (-\alpha x^3(t) + \beta \sin x(t))dt + \sigma(1 - 2 \sin x(t))dW(t), \quad (6)$$

$$t \in [0, T], \quad x(0) = 0; \quad \alpha, \beta, \sigma \in \mathbb{R}.$$

$$\text{Drift coefficient} \rightarrow a(x) = (-\alpha x^3 + \beta \sin x);$$

$$\text{Diffusion coefficient} \rightarrow b(x) = \sigma(1 - 2 \sin x).$$

- a, b - continuously-differentiable $\Rightarrow$ locally Lipschitz continuous
- a - not globally Lipschitz continuous
- b - globally Lipschitz continuous
- a - satisfies one-sided Lipschitz condition for $\alpha > 0$

$\Rightarrow$ All assumptions of the existence and uniqueness theorem are satisfied and the equation (6) has unique solution x which satisfies $x(0) = 0$ a.s.

- Assumptions $\mathcal{A}_3$ and $\mathcal{A}_4$ are satisfied

$\Rightarrow$ Lemma (Higham, Mao, Stuart (2002)) implies that

$$E \sup_{0 \leq t \leq T} |x(t)|^r \leq C (1 + E|x(0)|^r) \leq Q, \text{ for } r > 2$$

$\Rightarrow$ Condition $\mathcal{A}_5$ holds for the solution x

NOTICE: Equation (6) is not explicitly solvable.

- Subsegments $[t_k, t_{k+1}]$ of the partition (3) are transformed into $[0, t_{k+1} - t_k]$, $k \in \{0, 1, \dots, n-1\}$.
- Approximate equations (4) are constructed by Maclaurin approximations of functions a and b instead of Taylor approximations of those functions near the points $x(t_k)$.

- Time translation $t = t_k + u$, for $k \in \{0, \dots, n-1\}$, transforms the equation (6) into

$$d\tilde{X}(u) = (-\alpha\tilde{X}^3(u) + \beta\sin\tilde{X}(u))du + \sigma(1 - 2\sin\tilde{X}(u))d\tilde{W}(u), \quad (7)$$

$$u \in [0, t_{k+1} - t_k], \quad k \in \{0, \dots, n-1\}.$$

$k = 0 \mapsto$ the initial condition is $\tilde{X}(0) = 0$ a.s.

$k \in \{1, \dots, n-1\} \mapsto$ the initial conditions are determined successively as the values of the process $\tilde{X}(u)$ in the points $t_k - t_{k-1}$.

(I) Maclaurin approximations:

$a \mapsto$ up to the 3rd derivative; $b \mapsto$ up to the 2nd derivative

$$a(x) \approx -\frac{6\alpha + \beta}{6}x^3 + \beta x, \quad b(x) \approx -2\sigma x + \sigma, \quad (x \rightarrow 0)$$

- $\mathcal{A}_1, \mathcal{A}_2$ hold $\Leftrightarrow (\sup_x |a^{(4)}(x)| \leq |\beta|, \sup_x |b'''(x)| \leq 2|\sigma|)$

The approximate solution $\{\tilde{x}^n(u), u \in [0, T]\}$ is constructed successively by using the solutions of the equations

$$d\tilde{x}^n(u) = \left(-\frac{6\alpha + \beta}{6}(\tilde{x}^n(u))^3 + \beta\tilde{x}^n(u) \right) du + \sigma(1 - 2\tilde{x}^n(u))d\tilde{W}(u) \quad (8)$$

$$u \in [0, t_{k+1} - t_k], \quad k \in \{0, \dots, n-1\}.$$

$k = 0 \mapsto$ the initial condition is $\tilde{x}^n(0) = 0$ a.s.

$k \in \{1, \dots, n-1\} \mapsto$ the initial conditions are determined successively as the values of the process $\tilde{x}^n(u)$ in the points $t_k - t_{k-1}$.

Equations (8):

- Aren't explicitly solvable
- Coefficients do satisfy the same conditions as coefficients of the initial equation (6) for $\alpha \geq -\beta/6$ and

$$E \sup_{0 \leq u \leq T} |\tilde{x}^n(u)|^r \leq c_1 (1 + E|x(0)|^r) \leq Q, \text{ for } r > 2.$$

The time translation implies

$$dx^n(t) = \left(-\frac{6\alpha + \beta}{6} (x^n(t))^3 + \beta x^n(t) \right) dt + \sigma(1 - 2x^n(t)) dW(t) \quad (9)$$

$$t \in [t_k, t_{k+1}], \quad k \in \{0, \dots, n-1\}$$

Successive connecting of the processes $\{x^n(t), t \in [t_k, t_{k+1}]\}$, $k \in \{0, \dots, n-1\}$, in the partition points is almost surely continuous solution $\{x^n(t), t \in [0, T]\}$.

Theorem 2 gives the rate of closeness in the L^p sense

$$E \sup_{t \in [0, T]} |x(t) - x^n(t)|^p \leq K \delta_n^{\frac{3}{2}p}.$$

(II) Maclaurin approximations:

$a \mapsto$ up to the 2nd derivative; $b \mapsto$ up to the 2nd derivative

$$a(x) \approx \beta x, \quad b(x) \approx -2\sigma x + \sigma, \quad (x \rightarrow 0)$$

The approximate solution $\{\tilde{x}^n(u), u \in [0, T]\}$ is constructed using the equations

$$\begin{aligned} d\tilde{x}^n(u) &= \beta \tilde{x}^n(u) du + \sigma(1 - 2\tilde{x}^n(u)) d\tilde{W}(u), \\ u &\in [0, t_{k+1} - t_k], \quad k \in \{0, \dots, n-1\}. \end{aligned} \quad (10)$$

$k = 0 \mapsto$ the initial condition is $\tilde{x}^n(0) = 0$ a.s.

$k \in \{1, \dots, n-1\} \mapsto$ the initial conditions are determined successively as the values of the process $\tilde{x}^n(u)$ in the points $t_k - t_{k-1}$.

Equations (10) are inhomogeneous linear stochastic differential equations with multiplicative noise:

- Coefficients satisfy global Lipschitz condition and linear growth condition

⇒ Equations (10) are explicitly solvable (for example, Kloeden, Platen (1999)). Their solutions are

$$\begin{aligned}\tilde{x}^n(u) &= \tilde{x}_0^n e^{(\beta-2\sigma^2)u-2\sigma(\tilde{W}(u)-\tilde{W}(0))} \\ &\quad + 2\sigma^2 \int_0^u e^{-(\beta-2\sigma^2)(u-s)-2\sigma(\tilde{W}(u)-\tilde{W}(s))} ds \\ &\quad + \sigma \int_0^u e^{-(\beta-2\sigma^2)(u-s)+2\sigma(\tilde{W}(u)-\tilde{W}(s))} d\tilde{W}(s), \\ u &\in [0, t_{k+1} - t_k], \quad k \in \{0, \dots, n-1\},\end{aligned}$$

and, for $r > 2$, following holds (for example Mao (2008)),

$$E \sup_{0 \leq t \leq T} |x^n(t)|^r = E \sup_{0 \leq u \leq T} |\tilde{x}^n(u)|^r \leq c_2 (1 + 3^{r-1} E |x(0)|^r) \leq Q.$$

The time translation implies

$$\begin{aligned}
 x^n(t) &= x^n(t_k) e^{(\beta-2\sigma^2)(t-t_k)-2\sigma(W(t)-W(t_k))} \\
 &\quad + 2\sigma^2 \int_{t_k}^t e^{-(\beta-2\sigma^2)(t-s)-2\sigma(W(t)-W(s))} ds \\
 &\quad + \sigma \int_{t_k}^t e^{-(\beta-2\sigma^2)(t-s)-2\sigma(W(t)-W(s))} dW(s), \\
 t &\in [t_k, t_{k+1}], \quad k \in \{0, \dots, n-1\}.
 \end{aligned} \tag{11}$$

Successive connecting of the processes $\{x^n(t), t \in [t_k, t_{k+1}]\}$, $k \in \{0, \dots, n-1\}$, in the partition points, makes almost surely continuous solution $\{x^n(t), t \in [0, T]\}$. Theorem 2 gives the same rate of closeness in the L^p sense as in the previous case.

(III) Maclaurin approximations:

$a \mapsto$ up to the 2nd derivative; $b \mapsto$ up to the derivative zero

$$a(x) \approx \beta x, \quad b(x) \approx \sigma, \quad (x \rightarrow 0)$$

Then the equations

$$d\tilde{x}^n(u) = \beta \tilde{x}^n(u) du + \sigma d\tilde{W}(u), \quad u \in [0, t_{k+1} - t_k], \quad k \in \{0, \dots, n-1\},$$

are explicitly solvable (for example, Kloeden, Platen (1999)).

Their solutions are

$$\begin{aligned} \tilde{x}^n(u) &= e^{\beta u} \left(\tilde{x}_0^n + \sigma \int_0^u e^{-\beta s} d\tilde{W}(s) \right), \\ u &\in [0, t_{k+1} - t_k], \quad k \in \{0, \dots, n-1\}. \end{aligned}$$

Time translation implies

$$x^n(t) = e^{\beta(t-t_k)} \left(x^n(t_k) + \sigma \int_{t_k}^t e^{-\beta(s-t_k)} dW(s) \right), \quad (12)$$

$$t \in [t_k, t_{k+1}], \quad k \in \{0, \dots, n-1\}.$$

Successive connecting of the processes $\{x^n(t), t \in [t_k, t_{k+1}]\}$, $k \in \{0, \dots, n-1\}$, in the partition points, constructs almost surely continuous solution $\{x^n(t), t \in [0, T]\}$.

Theorem 2 gives the rate of closeness in the L^p sense

$$E \sup_{t \in [0, T]} |x(t) - x^n(t)|^p \leq K \delta_n^{\frac{1}{2}p}.$$

The most commonly used method $\rightarrow$ numerical Euler-Maruyama (EM) method.

It is known that this method, which is based on Taylor expansion of zero degrees, has order $1/2$.

Comparison for the approximate solutions of the initial equation (6) obtained by:

- The numerical EM method, and
- Analytical method described here (III) for different values of the parameters α , β and σ .

The EM method: $X_k \approx x(t_k)$, where $X_0 = 0$,

$$X_{k+1} = X_k + (-\alpha X_k^3 + \beta \sin X_k)(t_{k+1} - t_k) + \sigma(1 - 2 \sin X_k)(W(t_{k+1}) - W(t_k)), \quad (13)$$

$$\bar{X}(t) = X_k + (t - t_k)(-\alpha X_k^3 + \beta \sin X_k) + \sigma(1 - 2 \sin X_k)(W(t) - W(t_k)), \quad t \in [t_k, t_{k+1}],$$

and EM solution is defined by $X(t) = \bar{X}(t)$ for $t \in [t_k, t_{k+1}]$.

The result of this comparison can be seen in Figures 1 and 2.

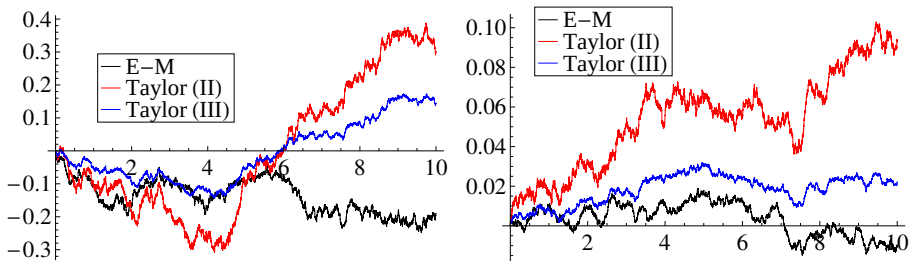


Figure: Trajectories of the solutions to the equations (13), (11) and (12) for $x(0) = 0$ and: $\alpha = 0.3$, $\beta = 0.3\pi^3/108$, $\sigma = 0.05$ (left), $\alpha = 0.1$, $\beta = 0.1$, $\sigma = 0.01$ (right), for $\delta_{10\,000} = 0.001$ on time interval $[0, 10]$

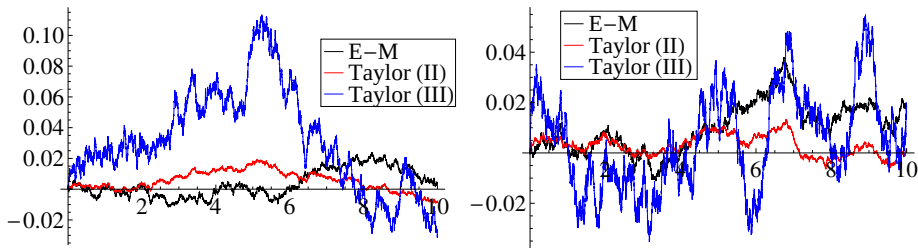


Figure: Trajectories of the solutions to the equations (13), (11) and (12) for $x(0) = 0$, $\alpha = 0.5$, $\beta = -0.2$ and: $\sigma = 0.01$ (left), $\sigma = -0.01$ (right), for $\delta_{10000} = 0.001$ on time interval $[0, 10]$

According to the Theorem 2, the sequence of the approximate solutions (11) has greater order of the L^p convergence comparing to solutions (12) and (13).

STOCHASTIC DIFFERENTIAL EQUATION WITH TIME DEPENDENT DELAY (DIFFERENTIAL FORM)

$$dx(t) = a(x(t), x(t - \delta(t)), t) dt + b(x(t), x(t - \delta(t)), t) dw(t), \\ t \in [t_0, T], \quad x_{t_0} = \eta = \{\eta(\theta), \theta \in [-\tau, 0]\} \quad (14)$$

- $0 \leq t_0 < T$ and $\tau > 0$ are fixed real numbers
- $a : \mathbb{R}^d \times \mathbb{R}^d \times [t_0, T] \rightarrow \mathbb{R}^d$ is the drift coefficient
- $b : \mathbb{R}^d \times \mathbb{R}^d \times [t_0, T] \rightarrow \mathbb{R}^{d \times d_1}$ is the diffusion coefficient
- $\delta : [t_0, T] \rightarrow [0, \tau]$ is the continuous delay function
- η is $\mathcal{F}_{t_0}$ –measurable, V –valued random variable

THEOREM (SUFFICIENT CONDITIONS FOR EU)

If the drift and diffusion coefficients of Eq. (14) satisfy both the (non-delay) local Lipschitz condition and linear growth condition, then there exists a unique solution $x = \{x(t), t \in [t_0 - \tau, T]\}$ to Eq. (14) with initial data $x_{t_0} = \eta$.



Consider SDEwTDD (14) of the Itô type in integral form:

STOCHASTIC DIFFERENTIAL EQUATION WITH TIME DEPENDENT DELAY (INTEGRAL FORM)

$$\begin{aligned}
 x(t) = & \eta(0) + \int_{t_0}^t a(x(u), x(u - \delta(u)), u) du \\
 & + \int_{t_0}^t b(x(u), x(u - \delta(u)), u) dw(u), \quad t \in [t_0, T]
 \end{aligned} \tag{15}$$

- Assume that τ and $T - t_0$ have a rational ratio $\Rightarrow$ for infinitely many positive integers n there exists positive integers n' such that $(T - t_0)/n = \tau/n' = \delta_n$
- n (n') is large enough such that $\delta_n < 1$
- Define the equidistant partition of the time interval $[t_0 - \tau, T]$:

$$t_0 - \tau = t_{-n'} < t_{-n'+1} < \cdots < t_{-1} < t_0 < t_1 < \cdots < t_n = T \tag{16}$$

NOTATIONS:

- For any $t \in [t_0, T]$: $\delta_1(t) = t - \delta(t)$
- For any fixed $t \in [t_i, t_{i+1}]$, $i \in \{0, \dots, n-1\}$,

$$S_i^j = \{s \in [t_i, t] \mid s - \delta(s) \in [t_j, t_{j+1} \wedge t)\}, \quad j \in \{i - n', \dots, i\}$$

- For an arbitrary $\mathcal{F}_t$ -adapted stochastic process $y^n = \{y^n(t) \mid t \in [t_0 - \tau, T]\}$, the new step-process $\tilde{y}^n = \{\tilde{y}^n(t) \mid t \in [t_0, T]\}$ is introduced as

$$\tilde{y}^n(s) = \sum_{j=i-n'}^i \chi_{S_i^j}(s) y^n(t_j), \quad s \in [t_i, t],$$

for any $t \in [t_i, t_{i+1}]$, $i \in \{0, \dots, n-1\}$

GOAL:

Approximation of the solution $x = \{x(t), t \in [t_0 - \tau, T]\}$ of Eq. (15) with initial condition η on partition (16) via processes noted as $x^n = \{x^n(t), t \in [t_0 - \tau, T]\}$, which are obtained by successive connecting of solutions of the following Eqs, for $k \in \{0, \dots, n-1\}$,

$$x^n(t) = x^n(t_k) + \int_{t_k}^t \sum_{i=0}^{m_1} \frac{d^i a(x^n(t_k), \tilde{x}^n(s), s)}{i!} ds \quad (17)$$

$$+ \int_{t_k}^t \sum_{i=0}^{m_2} \frac{d^i b(x^n(t_k), \tilde{x}^n(s), s)}{i!} dw(s), \quad t \in [t_k, t_{k+1}],$$

with the initial data $x_{t_0}^n = \eta$ for $k = 0$ and $x_{t_k}^n$ for $k \in \{1, \dots, n-1\}$, where $x_{t_k}^n = \{x^n(t), t \in [t_k - \tau, t_k]\}$.

The stochastic process $x^n = \{x^n(t), t \in [t_0 - \tau, T]\}$ obtained this way is almost surely continuous process.

- Eqs. (17) have drift and diffusion coefficients which are Taylor expansions up to the m_1 -th and m_2 -th derivatives, respectively, in the first two arguments of the drift and diffusion coefficients of initial Eq. (15).
- In Eqs. (17), $d^i a(x^n(t_k), \tilde{x}^n(s), s)$ and $d^i b(x^n(t_k), \tilde{x}^n(s), s)$, for $s \in [t_k, t_{k+1}]$, $k \in \{0, \dots, n-1\}$, represent $a(x^n(t_k), \tilde{x}^n(s), s) \in \mathbb{R}^d$ and $b(x^n(t_k), \tilde{x}^n(s), s) \in \mathbb{R}^{d_1 \times d}$, respectively, for $i = 0$ and for $i \geq 1$ they are

$$d^i a(x^n(t_k), \tilde{x}^n(s), s) = \sum_{j=0}^i \binom{i}{j} \frac{\partial^i a(x^n(t_k), \tilde{x}^n(s), s)}{\partial x^j \partial y^{i-j}} [x^n(s) - x^n(t_k)]^j [x^n(\delta_1(s)) - \tilde{x}^n(s)]^{i-j},$$

$$d^i b(x^n(t_k), \tilde{x}^n(s), s) = \sum_{j=0}^i \binom{i}{j} \frac{\partial^i b(x^n(t_k), \tilde{x}^n(s), s)}{\partial x^j \partial y^{i-j}} [x^n(s) - x^n(t_k)]^j [x^n(\delta_1(s)) - \tilde{x}^n(s)]^{i-j}.$$

ASSUMPTIONS:

$\mathcal{A}'_1$: There exist Taylor expansions of the functionals a and b in the first two arguments up to the m_1 -th and m_2 -th derivatives, respectively.

$\mathcal{A}'_2$: All the partial Fréchet derivatives with respect to the first two arguments of the order $m_1 + 1$ for the functional a and of the order $m_2 + 1$ for the functional b are uniformly bounded, that is, there exist positive real constants $\bar{L}_a$ and $\bar{L}_b$ such that

$$\sup_{(x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times [t_0, T]} \left\| \frac{\partial^{m_1+1} a_{(x,y,t)}}{\partial x^j \partial y^{m_1+1-j}} \right\|_{m_1+1} \leq \bar{L}_a, \quad j \in \{0, \dots, m_1 + 1\},$$

$$\sup_{(x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times [t_0, T]} \left\| \frac{\partial^{m_2+1} b_{(x,y,t)}}{\partial x^j \partial y^{m_2+1-j}} \right\|_{m_2+1} \leq \bar{L}_b, \quad j \in \{0, \dots, m_2 + 1\}.$$

REMARK:

Boundedness of the partial derivatives in first $2d$ arguments of coordinate functions of functionals a and b of the orders $m_1 + 1$ and $m_2 + 1$, respectively, is sufficient for $\mathcal{A}'_2$:

If there exist positive constants L_a and L_b such that for every $i \in \{1, \dots, d\}$, $k \in \{1, \dots, d \times d_1\}$

$$\sup_{(x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times [t_0, T]} \left| \frac{\partial^{m_1+1} a_i(x, y, t)}{\underbrace{\partial x_1 \dots \partial x_d}_j \underbrace{\partial y_{d+1} \dots \partial y_{2d}}_{m_1+1-j}} \right| \leq L_a, \quad j \in \{0, \dots, m_1 + 1\},$$

$$\sup_{(x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times [t_0, T]} \left| \frac{\partial^{m_2+1} b_k(x, y, t)}{\underbrace{\partial x_1 \dots \partial x_d}_j \underbrace{\partial y_{d+1} \dots \partial y_{2d}}_{m_2+1-j}} \right| \leq L_b, \quad j \in \{0, \dots, m_2 + 1\},$$

$a_i, b_k : \mathbb{R}^d \times \mathbb{R}^d \times [t_0, T] \rightarrow \mathbb{R}$, $i \in \{1, \dots, d\}$, $k \in \{1, \dots, d \times d_1\}$,
 $a = (a_1, \dots, a_d)$, $b = (b_1, \dots, b_{d \times d_1})$, then $\mathcal{A}'_2$ holds.

$\mathcal{A}'_3$: Functionals a and b satisfy the polynomial condition: there exist positive real number $\bar{D}$ and a nonnegative integer $\bar{q}$ such that for every $x, \tilde{x}, y, \tilde{y} \in \mathbb{R}^d$ and $t \in [t_0, T]$,

$$\begin{aligned} & |a(x, y, t) - a(\tilde{x}, \tilde{y}, t)|^2 \vee |b(x, y, t) - b(\tilde{x}, \tilde{y}, t)|^2 \\ & \leq \bar{D} \left(1 + |(x, y)|^{\bar{q}} + |(\tilde{x}, \tilde{y})|^{\bar{q}} \right) |(x, y) - (\tilde{x}, \tilde{y})|^2. \end{aligned}$$

$\mathcal{A}'_4$: Functions $a(0, 0, \cdot)$ and $b(0, 0, \cdot)$ are bounded on $[t_0, T]$, that is, there exist positive constants $\bar{K}_a$ and $\bar{K}_b$ such that $|a(0, 0, t)| \leq \bar{K}_a$ and $|b(0, 0, t)| \leq \bar{K}_b$, for $t \in [t_0, T]$.

$\mathcal{A}'_5$: There exist unique solutions x and x^n of Eqs. (15) and (17), respectively, with appropriate initial data η such that for some $\bar{p} \geq 2$ and $\bar{Q} \geq 0$ independent of n , for $M = m_1 \vee m_2$,

$$E \sup_{t \in [t_0 - \tau, T]} |x(t)|^{2\bar{p}(1 \vee \bar{q})} \leq \bar{Q} < \infty,$$

$$E \sup_{t \in [t_0 - \tau, T]} |x^n(t)|^{\bar{p}[2 \vee 2\bar{q} \vee (1 + \bar{q}/2)(M+1) \vee (M+1)^2]} \leq \bar{Q} < \infty.$$

$\mathcal{B}_6$: Stochastic process η satisfies the polynomial condition, that is, there exist positive constant D' and a nonnegative integer q' such that for every two real numbers $\theta_1, \theta_2 \in [-\tau, 0]$,

$$|\eta(\theta_1) - \eta(\theta_2)|^2 \leq D'(1 + |\theta_1|^{q'} + |\theta_2|^{q'})|\theta_1 - \theta_2|^2.$$

NOTE:

- Delay function δ is supposed to be continuous, but usually authors have more restrictive assumption: Lipschitz continuity.
- Also, the initial data η is usually Lipschitz continuous and we assumed more general condition: the polynomial condition.

Lemma

I Let $\{x^n(t), t \in [t_k, t_{k+1}]\}$, $k \in \{0, \dots, n-1\}$ be the solutions of Eqs. (17) and let us assume that the assumptions $\mathcal{A}'_1 - \mathcal{A}'_5$ are satisfied. Then, for every $0 < r \leq (M+1)\bar{p}$,

$$E \sup_{s \in [t_k, t]} |x^n(s) - x^n(t_k)|^r \leq \bar{C}' \delta_n^{r/2}, \quad t \in [t_k, t_{k+1}], \quad k \in \{0, \dots, n-1\},$$

where $\bar{C}'$ represents a positive constant independent of n .

II In addition to the given assumptions, let $\mathcal{B}_6$ be satisfied. Then

$$\sup_{s \in [t_k, t]} E |x^n(\delta_1(s)) - \tilde{x}^n(s)|^r \leq \bar{C}'' \delta_n^{r/2}, \quad t \in [t_k, t_{k+1}], \quad k \in \{0, \dots, n-1\},$$

where $\bar{C}''$ stands for a positive constant independent of n .

Theorem

Let all of the assumptions from Lemma 4 Part II hold and let x be the solution of Eq. (15) with the initial condition η . Then,

$$E \sup_{t \in [t_0 - \tau, T]} |x(t) - x^n(t)|^{\bar{p}} \leq \bar{C} \delta_n^{(m+1)\bar{p}/2},$$

where $m = m_1 \wedge m_2$ and $\bar{C}$ is a constant independent of n .

Theorem

Let the assumptions of Theorem 5 be satisfied. Then the sequence $(x^n)_{n \in \mathbb{N}}$ of the solutions to Eqs. (17) converges almost surely to the solution x of Eq. (15).

APPENDIX:

- The main reason for employing more restrictive assumption (Lipschitz continuity) on δ than the assumption we used (continuity) is **to determine the points** in which the derivatives of the coefficients of Eq. (15) with respect to the second argument will be defined and which will be fixed on each partitioning interval.
- These points are of the form $x^n(t_{k-\lfloor \delta(t_k)/\delta_n \rfloor})$ (instead of $\tilde{x}^n(s)$ for $s \in [t_k, t_{k+1}]$), $k \in \{0, \dots, n-1\}$.
- Previous Lemma and Theorems also hold under this, more restrictive assumption.

STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATION

$$\begin{aligned} dx(t) &= a(x_t, t)dt + b(x_t, t)dw(t), \quad t \in [t_0, T], \\ x_{t_0} &= \eta = \{\eta(\theta), \theta \in [-\tau, 0]\}. \end{aligned} \quad (18)$$

- $a : V \times [t_0, T] \rightarrow \mathbb{R}^d$ is the drift coefficient
- $b : V \times [t_0, T] \rightarrow \mathbb{R}^{d \times d_1}$ is the diffusion coefficient
- η is $\mathcal{F}_{t_0}$ -measurable, V -valued random variable
- $x_t = \{x(t + \theta) \mid \theta \in [-\tau, 0]\}$ is a stochastic process, $t \in [t_0, T]$
- $x_t(\theta) = x(t + \theta)$ for any $\theta \in [-\tau, 0]$ and $t \in [t_0, T]$
- $x(t)$ is an $\mathbb{R}^d$ -valued random variable for any $t \in [t_0 - \tau, T]$

THEOREM (SUFFICIENT CONDITIONS FOR EU)

If the drift and diffusion coefficients of Eq. (18) satisfy the local Lipschitz condition and the linear growth condition, then there exists a unique solution $x = \{x(t), t \in [t_0 - \tau, T]\}$ to Eq. (18) with initial data $x_{t_0} = \eta$.

Consider SFDE (18) of the Itô type in integral form:

STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATION (INTEGRAL FORM)

$$x(t) = \eta(0) + \int_{t_0}^t a(x_u, u) du + \int_{t_0}^t b(x_u, u) dw(u), \quad t \in [t_0, T] \quad (19)$$

- Define the equidistant partition of the time interval $[t_0, T]$:

$$t_0 < t_1 < \cdots < t_n = T, \quad \delta_n = \frac{T - t_0}{n} < 1. \quad (20)$$

GOAL:

Approximation of the solution $x = \{x(t), t \in [t_0 - \tau, T]\}$ of Eq. (19) with initial condition η on partition (20) via processes noted as $x^n = \{x^n(t), t \in [t_0 - \tau, T]\}$, which are obtained by successive connecting of solutions of the Eqs.

$$\begin{aligned}
 x^n(t) = & x^n(t_k) + \int_{t_k}^t \sum_{i=0}^{m_1} \frac{a_{(x_{t_k}^n, s)}^{(i)} (x_s^n - x_{t_k}^n)^i}{i!} ds \\
 & + \int_{t_k}^t \sum_{i=0}^{m_2} \frac{b_{(x_{t_k}^n, s)}^{(i)} (x_s^n - x_{t_k}^n)^i}{i!} dw(s), \quad t \in [t_k, t_{k+1}],
 \end{aligned} \tag{21}$$

with the initial data $x_{t_0}^n = \eta$ for $k = 0$ and $x_{t_k}^n$ for $k \in \{1, \dots, n-1\}$, in the division points of partition (20).

ASSUMPTIONS:

$\mathcal{A}_1''$: There exist Taylor expansions of the functionals a and b in the first argument up to the m_1 -th and m_2 -th Fréchet derivatives, respectively.

$\mathcal{A}_2''$: Functionals $a^{(m_1+1)}$ and $b^{(m_2+1)}$ are bounded in the following way: $(\exists L_a, L_b > 0) (\forall h \in V)$

$$\sup_{(x,t) \in V \times [t_0, T]} |a_{(x,t)}^{(m_1+1)}(h, \dots, h)| \leq L_a \|h\|^{m_1+1},$$

$$\sup_{(x,t) \in V \times [t_0, T]} |b_{(x,t)}^{(m_2+1)}(h, \dots, h)| \leq L_b \|h\|^{m_2+1}.$$

$\mathcal{A}_3''$ Functionals a and b satisfy the polynomial condition:

$$(\exists D > 0) (\exists q \in \mathbb{N}_0) (\forall x, y \in V) (\forall t \in [t_0, T])$$

$$|a(x, t) - a(y, t)|^2 \vee |b(x, t) - b(y, t)|^2 \leq D(1 + \|x\|^q + \|y\|^q) \|x - y\|^2.$$

$\mathcal{A}_4''$: Functions $a(0, \cdot)$ and $b(0, \cdot)$ are bounded on $[t_0, T]$:

$$(\exists K_a, K_b < \infty) (\forall t \in [t_0, T]) \quad |a(0, t)| \leq K_a, \quad |b(0, t)| \leq K_b.$$

$\mathcal{A}_5''$: There exist unique solutions x and x^n of Eqs. (19) and (21), respectively, for appropriate initial data η such that for some $p \geq 2$ and $Q \geq 0$ independent of n , for $M = m_1 \vee m_2$,

$$E \sup_{t \in [t_0 - \tau, T]} |x(t)|^{2p(1 \vee q)} \leq Q < \infty,$$

$$E \sup_{t \in [t_0 - \tau, T]} |x^n(t)|^{p[2 \vee 2q \vee (M+1)(1+q/2) \vee (M+1)^2]} \leq Q < \infty.$$

$\mathcal{C}_6$: Stochastic process η satisfies the polynomial condition, that is, there exist positive constant D' and a nonnegative integer q' such that for every two real numbers $\theta_1, \theta_2 \in [-\tau, 0]$,

$$|\eta(\theta_1) - \eta(\theta_2)|^2 \leq D'(1 + |\theta_1|^{q'} + |\theta_2|^{q'})|\theta_1 - \theta_2|^2.$$

Lemma

Let $\{x^n(t), t \in [t_k - \tau, t_{k+1}]\}$, $k \in \{0, \dots, n-1\}$ represent the solutions of Eqs. (21) with the appropriate initial conditions and let us assume that the conditions $\mathcal{A}_1'' - \mathcal{A}_5''$ are satisfied. Then, for every $0 < r \leq (M+1)p$,

$$E \sup_{s \in [t_k, t]} |x^n(s) - x^n(t_k)|^r \leq C' \delta_n^{r/2}, \quad t \in [t_k, t_{k+1}], \quad k \in \{0, \dots, n-1\},$$

where C' is a constant independent n .

- Equidistance of the partition (20) does not play any role in the proof of the previous Lemma, but it is required in the proof of following lemma.

Lemma

Let the conditions of Lemma 7 and assumption $\mathcal{C}_6$ be satisfied. Then, for every $0 < r \leq (M + 1)p$,

$$E \|x_t^n - x_{t_k}^n\|^r \leq C'' \delta_n^{r/2}, \quad t \in [t_k, t_{k+1}], \quad k \in \{0, \dots, n-1\},$$

where C'' is a positive constant independent of n .

Theorem

Let us assume that all assumptions of Lemma 8 hold, let x be the solution of Eq. (19) with the initial condition η and x^n be the solutions of Eqs. (21). Then, for every $p \geq 2$ which satisfies $\mathcal{A}_5''$,

$$E \sup_{t \in [t_0 - \tau, T]} |x(t) - x^n(t)|^p \leq C \delta_n^{(m+1)p/2},$$

where $m = m_1 \wedge m_2$ and C is a constant independent of n .

Theorem

Let the assumptions of Theorem 9 be satisfied. Then, the sequence $(x^n)_{n \in \mathbb{N}}$ of the solutions of Eqs. (21) converges almost surely to the solution x of Eq. (19).

ASSUMPTIONS:

$\mathcal{D}_2$: Functions $a_{(0,\cdot)}^{(m_1+1)}$ and $b_{(0,\cdot)}^{(m_2+1)}$ are bounded on $[t_0, T]$, that is, there exist positive constants K'_a and K'_b such that

$$\|a_{(0,t)}^{(m_1+1)}\|_{m_1+1} \leq K'_a \text{ and } \|b_{(0,t)}^{(m_2+1)}\|_{m_2+1} \leq K'_b, \text{ for } t \in [t_0, T].$$

$\mathcal{D}_3$: Functionals $a, b, a^{(m_1+1)}, b^{(m_2+1)}$ satisfy the following polynomial conditions: $(\exists D_a, D_b, D'_a, D'_b > 0)(\exists q_a, q_b, q'_a, q'_b \geq 0)$
 $(\forall x, y \in \mathbb{R}^d)(\forall t \in [t_0, T]),$

$$|a(x, t) - a(y, t)|^2 \leq D_a(1 + \|x\|^{q_a} + \|y\|^{q_a})\|x - y\|^2,$$

$$|b(x, t) - b(y, t)|^2 \leq D_b(1 + \|x\|^{q_b} + \|y\|^{q_b})\|x - y\|^2,$$

$$\|a_{(x,t)}^{(m_1+1)} - a_{(y,t)}^{(m_1+1)}\|_{m_1+1}^2 \leq D'_a(1 + \|x\|^{q'_a} + \|y\|^{q'_a})\|x - y\|^2,$$

$$\|b_{(x,t)}^{(m_2+1)} - b_{(y,t)}^{(m_2+1)}\|_{m_2+1}^2 \leq D'_b(1 + \|x\|^{q'_b} + \|y\|^{q'_b})\|x - y\|^2.$$

$\mathcal{D}_5$: There exist unique solutions x and x^n of Eqs. (19) and (21), respectively, for appropriate initial data η such that for some $p \geq 2$ and $Q \geq 0$ independent of n , for $M = m_1 \vee m_2$,

$$E \sup_{t \in [t_0 - \tau, T]} |x(t)|^{2p(1 \vee q)} \leq Q < \infty,$$

$$E \sup_{t \in [t_0 - \tau, T]} |x^n(t)|^{2p[1+(M \vee (q/2))][2+M+(M \vee (q/2))]} \leq Q < \infty.$$

NOTE:

- The initial data η is usually Lipschitz continuous and we assumed more general condition: the polynomial condition.
- Equidistance of the partition (20) does not play any role in the proof of the following lemma.

Lemma

Let $\{x^n(t), t \in [t_k - \tau, t_{k+1}]\}$, $k \in \{0, \dots, n-1\}$ represent the solutions of Eqs. (21) with the initial condition η and let us assume that the conditions $\mathcal{A}_1'', \mathcal{A}_4'', \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_5$ are satisfied. Then, for every $0 < r \leq (M + 2 + [M \vee \frac{q}{2}])p$,

$$E \sup_{s \in [t_k, t]} |x^n(s) - x^n(t_k)|^r \leq C \delta_n^{r/2}, \quad t \in [t_k, t_{k+1}], \quad k \in \{0, \dots, n-1\},$$

where C is a constant independent of both n and δ_n .

Lemma

Let the conditions of Lemma 11 and assumption $\mathcal{C}_6$ be satisfied. Then, for every $0 < r \leq (M + 2 + [M \vee \frac{q}{2}])p$,

$$E \|x_t^n - x_{t_k}^n\|^r \leq \bar{B} \delta_n^{r/2}, \quad t \in [t_k, t_{k+1}], \quad k \in \{0, \dots, n-1\},$$

where $\bar{B}$ is a positive constant independent of both n and δ_n .

Theorem

Let us assume that all assumptions of Lemma 12 hold, let x be the solution of Eq. (19) with the initial condition η and x^n be the solutions of Eqs. (21). Then, for $p > 0$

$$E \sup_{t \in [t_0 - \tau, T]} |x(t) - x^n(t)|^p \leq H \delta_n^{(m+1)p/2},$$

where $m = m_1 \wedge m_2$ and H is a constant independent of n and δ_n .

Theorem

Let the assumptions of Theorem 13 be satisfied. Then, the sequence $(x^n)_{n \in \mathbb{N}}$ of the solutions of Eqs. (21) converges almost surely to the solution x of Eq. (19).

EXAMPLE

ONE-DIMENSIONAL ($d = 1$) SFDE

$$dx(t) = \left[-\alpha(t)C(x(t)) + 0.5\beta(t) \sin \left(2 \int_{-\tau}^0 K(\theta)x_t(\theta)d\theta \right) \right] dt + a_1(t) \sin \left(\int_{-\tau}^0 K(\theta)x_t(\theta)d\theta \right) dw(t), \quad t \in [t_0, T]$$

(22)

- Initial condition $x_{t_0} = \eta = \{\theta, \theta \in [-\tau, 0]\}$ - independent of the one-dimensional ($d_1 = 1$) Brownian motion w
- $C(x) = \frac{1}{2} \int_0^{x^2} \frac{\cos y}{\sqrt{y}} dy$, $x \in \mathbb{R}$ is the Fresnel integral
- $\alpha, \beta, a_1 : [t_0, T] \rightarrow \mathbb{R}$ - nonnegative, continuous ($\Rightarrow$ bounded)
- Boundaries: $(\forall t \in [t_0, T]) (\alpha(t) \leq \bar{\alpha}, \beta(t) \leq \bar{\beta}, a_1(t) \leq \bar{a}_1)$.

- The drift and diffusion coefficients from (22) are considered for **an arbitrary but fixed** argument $t \in [t_0, T]$.
- These coefficients can be represented as

$$a_t \equiv a = f \circ \underline{h} + A_1 : V \rightarrow \mathbb{R}, \quad b_t \equiv b = g \circ \underline{h} : V \rightarrow \mathbb{R}$$

- $\underline{h} : V \rightarrow \mathbb{R}$ is defined as

$$\underline{h}(v) = \int_{-\tau}^0 K(\theta) v(\theta) d\theta, \quad v \in V$$

- $K : [-\tau, 0] \rightarrow \mathbb{R}_+$ - continuous, satisfies $\int_{-\tau}^0 K(\theta) d\theta = 1$ (Boundedness is enough). For the purposes of this example, we choose

$$K(\theta) = \frac{1}{\tau}, \quad \theta \in [-\tau, 0]$$

- Functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are defined as

$$f(x) = 0.5\beta(t) \sin(2x), \quad g(x) = a_1(t) \sin x$$

- $A_1 : V \rightarrow \mathbb{R}$ is defined as

$$A_1(v) = -\alpha(t)C(v(0))$$

Coefficients a and b from (22) are continuously-differentiable (in the first argument) and globally Lipschitz continuous:

$$(\exists L_a = 2(\bar{\alpha}^2 + \bar{\beta}^2) \geq 0, L_b = \bar{a}_1^2 \geq 0)(\forall u, v \in V)(\forall t \in [t_0, T])$$

$$|a(u, t) - a(v, t)|^2 \leq L_a \|u - v\|^2, \quad |b(u, t) - b(v, t)|^2 \leq L_b \|u - v\|^2.$$

FIRST FRÉCHET DERIVATIVES:

Functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are differentiable arbitrary many times in any $x \in \mathbb{R}$, so, there exist Fréchet derivatives $f'_x, g'_x \in \mathcal{L}(\mathbb{R}, \mathbb{R})$:
 $(\forall y \in \mathbb{R})(f'_x(y) = f'(x) \cdot y \in \mathbb{R}, g'_x(y) = g'(x) \cdot y \in \mathbb{R}).$

Obviously $\underline{h} \in \mathcal{L}(V, \mathbb{R})$, so, for every $u \in V$, there exists the first Fréchet derivative in u : $\underline{h}'_u = \underline{h} \in \mathcal{L}(V, \mathbb{R})$.

Also, there exists the second Fréchet derivative in u :
 $\underline{h}''_u = \mathbf{0} \in \mathcal{L}(V, \mathcal{L}(V, \mathbb{R})) \cong \mathcal{T}(V^2, \mathbb{R})$.

- Let $u \in V$ be arbitrary. To find A'_{1u} explicitly, if it exists, we make the difference

$$A_1(u + h) - A_1(u) = L_{A_1}h + R_{A_1}(u, h), \quad h \in V.$$

- If we can make this decomposition such that $|R_{A_1}(u, h)|/\|h\|_V \rightarrow 0$ as $\|h\|_V \rightarrow 0$ and $L_{A_1} \in \mathcal{L}(V, \mathbb{R})$, then $L_{A_1} = A'_{1u}$.

There exists the first Fréchet derivative of A_1 in arbitrary $u \in V$, $A'_{1u} \in \mathcal{L}(V, \mathbb{R})$: $A'_{1u}(v) = -\alpha(t) \cos(u^2(0))v(0)$.

The chain rule (for example Flett (1980)) implies that there exists first Fréchet derivative of a in any $u \in V$,

$$a'_u = (f \circ \underline{h} + A_1)'_u = f'_{\underline{h}(u)} \underline{h}'_u + A'_{1u} = f'_{\underline{h}(u)} \underline{h} + A'_{1u} \in \mathcal{L}(V, \mathbb{R}).$$

- This means that, for every $v \in V$

$$\begin{aligned} a'_u(v) &= f'(\underline{h}(u)) \underline{h}(v) + A'_{1u}(v) \\ &= -\alpha(t) \cos(u^2(0)) v(0) + \beta(t) \cos\left(2 \int_{-\tau}^0 K(\theta) u(\theta) d\theta\right) \int_{-\tau}^0 K(\theta) v(\theta) d\theta. \end{aligned} \quad (23)$$

Also, $b'_u = g'_{\underline{h}(u)} \underline{h} \in \mathcal{L}(V, \mathbb{R})$ in every $u \in V$. In other words,

$$b'_u(v) = g'(\underline{h}(u)) \underline{h}(v) = a_1(t) \cos\left(\int_{-\tau}^0 K(\theta) u(\theta) d\theta\right) \int_{-\tau}^0 K(\theta) v(\theta) d\theta. \quad (24)$$

SECOND FRÉCHET DERIVATIVES:

- $\exists a''_u \in \mathcal{L}(V, \mathcal{L}(V, \mathbb{R})) \iff V \ni v \xrightarrow{c} a'_v \in \mathcal{L}(V, \mathbb{R})$ is Fréchet differentiable in $u \in V$.
- To find a''_u explicitly, we make the difference

$$c(u+h) - c(u) = L_c h + R_c(u, h), \quad h \in V.$$

- If we can make this decomposition such that $\|R_c(u, h)\|_{\mathcal{L}(V, \mathbb{R})} / \|h\|_V \rightarrow 0$ as $\|h\|_V \rightarrow 0$ and $L_c \in \mathcal{L}(V, \mathcal{L}(V, \mathbb{R}))$, then $L_c = c'_u = a''_u \in \mathcal{L}(V, \mathcal{L}(V, \mathbb{R}))$.

It turns out that for every $v_1, v_2 \in V$,

$$\begin{aligned} a''_u(v_1)(v_2) &= 2\alpha(t)u(0) \sin u^2(0)v_1(0)v_2(0) \\ &\quad - 2\beta(t) \sin(2\underline{h}(u))\underline{h}(v_1)\underline{h}(v_2) \in \mathbb{R}, \\ b''_u(v_1)(v_2) &= -a_1(t) \sin(\underline{h}(u))\underline{h}(v_1)\underline{h}(v_2) \in \mathbb{R}. \end{aligned}$$

- The polynomial condition $\mathcal{D}_3$ holds for a , b , a'' and b'' with
 $q'_a = 4$, $D'_a = 16[(\bar{\alpha}^2 + 2\bar{\beta}^2) \vee 3\bar{\alpha}^2]$; $q'_b = 0$, $D'_b = \bar{a}_1^2$;
 $q_a = 0$, $D_a = 2(\bar{\alpha}^2 + \bar{\beta}^2)$; $q_b = 0$, $D_b = \bar{a}_1^2$.
- Conditions $\mathcal{A}_4''$ and $\mathcal{D}_2$ hold for arbitrary $K_a, K_b, K'_a, K'_b > 0$.
- The polynomial condition $\mathcal{C}_6$ for initial data η is satisfied with the constants $D' = 1$ and $q' = 0$.
- Condition $\mathcal{D}_5$ holds for the solution x of Eq. (22) (for example, Mao (2008)).

The second derivative of the drift coefficient a of Eq. (22) is not uniformly bounded, so, the L^p and a.s. convergence results from Milošević, Jovanović, Janković (2010) can not be applied.

We will prove the existence, uniqueness and p -th moment boundedness (for any $p \geq 2$) of the solution to the corresponding approximate equation.

- Using the Taylor expansion of the first order ($m_1 = m_2 = 1$) near the points $x^n(t_k)$ for $k \in \{0, \dots, n-1\}$, Eqs. (21) become

$$x^n(t) = x^n(t_k) + \int_{t_k}^t A(x_s^n, s; x_{t_k}^n) ds + \int_{t_k}^t B(x_s^n, s; x_{t_k}^n) dw(s), \quad (25)$$

$$t \in [t_k, t_{k+1}], \quad k \in \{0, 1, \dots, n-1\}.$$

- The drift and diffusion coefficients are

$$\begin{aligned} A(x_t^n, t; x_{t_k}^n) &= -\alpha(t)C(x^n(t_k)) + 0.5\beta(t) \sin(2h(x_{t_k}^n)) \\ &\quad - \alpha(t) \cos(x^n(t_k))^2 [x^n(t) - x^n(t_k)] \\ &\quad + \beta(t) \cos(2h(x_{t_k}^n)) [\underline{h}(x_t^n) - \underline{h}(x_{t_k}^n)], \\ B(x_t^n, t; x_{t_k}^n) &= a_1(t) \sin \underline{h}(x_{t_k}^n) + a_1(t) \cos \underline{h}(x_{t_k}^n) [\underline{h}(x_t^n) - \underline{h}(x_{t_k}^n)]. \end{aligned} \quad (26)$$

NOTATION:

$$z^n(t+\theta) = \sum_{k=0}^{n-1} I_{[t_k, t_{k+1})}(t) x^n(t_k + \theta), \quad \theta \in [-\tau, 0], \quad t \in [t_0, T] \quad (27)$$

- Eqs. (25) can be represented as

$$x^n(t) = \eta(0) + \int_{t_0}^t A(s, x_s^n, z_s^n) ds + \int_{t_0}^t B(s, x_s^n, z_s^n) dw(s), \quad t \in [t_0, T] \quad (28)$$

- For any $s \in [t_0, T]$,

$$A(s, x_s^n, z_s^n) = -\alpha(s) C(z^n(s)) - \alpha(s) \cos(z^n(s))^2 (x^n(s) - z^n(s)) \quad (29)$$

$$+ \frac{1}{2} \beta(s) \sin(2h(z_s^n)) + \beta(s) \cos(2h(z_s^n)) (\underline{h}(x_s^n) - \underline{h}(z_s^n)),$$

$$B(s, x_s^n, z_s^n) = a_1(s) \sin(\underline{h}(z_s^n)) + a_1(s) \cos(\underline{h}(z_s^n)) (\underline{h}(x_s^n) - \underline{h}(z_s^n)). \quad (30)$$

- We use the Khasminskii approach (this approach is successfully applied in many papers, such as, for example, Song, Hu, Mao, Zhang (2013) in the context of SFDEs) to prove the existence, uniqueness and moment boundedness of the solution to Eq. (28).
- Coefficients of Eq. (28) satisfy the local Lipschitz condition.

So, for the initial condition η there exists a unique local solution $\{x(t), t \in [t_0 - \tau, \tau_e)\}$, where τ_e is the explosion time.

- There exists an integer r_0 such that

$$\|\eta\| = \sup_{\theta \in [-\tau, 0]} |\eta(\theta)| = \tau < r_0.$$

- For each integer $r \geq r_0$, define the stopping time

$$\tau_r = T \wedge \inf\{t \in [t_0, \tau_e) : |x^n(t)| \geq r\}, \quad \inf \emptyset = \infty. \quad (31)$$

- Obviously, τ_r increases as $r \rightarrow \infty$.

- Define $\tau_\infty = \lim_{r \rightarrow \infty} \tau_r$, $\tau_\infty \leq \tau_e$ a.s.
- GOAL: prove that $\tau_\infty = T$ a.s., which implies that $\tau_e = T$,
 (the solution $\{x(t), t \in [t_0 - \tau, T]\}$ does not explode in finite time)

Let us assume that $p \geq 2$. Using the Ito formula,
 Burkholder-Davis-Gundy and Young inequality, we obtain

$$\begin{aligned}
 & E \sup_{s \in [t_0 - \tau, t]} |x^n(s \wedge \tau_r)|^p \\
 & \leq 2E \|\eta\|^p + 3p(11p - 21) \int_{t_0}^t E \sup_{u \in [t_0 - \tau, s]} |x^n(u \wedge \tau_r)|^p ds \\
 & \quad + 2E \int_{t_0}^{t \wedge \tau_r} |A(s, x_s^n, z_s^n)|^p ds + 2(33p - 1)E \int_{t_0}^{t \wedge \tau_r} |B(s, x_s^n, z_s^n)|^p ds.
 \end{aligned} \tag{32}$$

- The Fresnel integral is bounded by the constant 0.977451424
- There exists $C_1 > 0$ such that $C_1^p \geq 4^{p-1} \{ [\bar{\alpha}^p + (\bar{\beta}/2)^p] \vee \bar{\beta}^p \}$

$$|A(s, x_s^n, z_s^n)|^p \leq C_1^p [1 + |x^n(s) - z^n(s)|^p + |\underline{h}(x_s^n) - \underline{h}(z_s^n)|^p] \quad (33)$$

- There exists $C_2 > 0$ such that $C_2^p \geq 2^{p-1} \bar{a}_1^p$

$$|B(s, x_s^n, z_s^n)|^p \leq C_2^p [1 + |\underline{h}(x_s^n) - \underline{h}(z_s^n)|^p]. \quad (34)$$

- By substituting (33) and (34) into (32), with the application of the Hölder inequality, we have that, for all $t \in [t_0, T]$,

$$E \sup_{s \in [t_0 - \tau, t]} |x^n(s \wedge \tau_r)|^p \leq H_1(p) + H_2(p) \int_{t_0}^t E \sup_{u \in [t_0 - \tau, s]} |x^n(u \wedge \tau_r)|^p ds, \quad (35)$$

where

$$H_1(p) = 2E\|\eta\|^p + 2\left[C_1^p + (33p - 1)C_2^p\right](T - t_0),$$

$$H_2(p) = 3p(11p - 21) + 2^{p+1}C_1^p + 2^p(33p - 1)C_2^p.$$

- The Gronwall–Bellman lemma yields

$$E \sup_{s \in [t_0 - \tau, t]} |x^n(s \wedge \tau_r)|^p \leq H_1(p)e^{H_2(p)(T - t_0)}, \quad r \geq r_0, \quad t \in [t_0, T].$$

- Particularly, we have that

$$E \sup_{s \in [t_0 - \tau, T]} |x^n(s \wedge \tau_r)|^p \leq H_1(p) e^{H_2(p)(T-t_0)}, \quad r \geq r_0. \quad (36)$$

- In a view of (31) and (36), we have that

$$r^p P\{\tau_r \leq T\} \leq H_1(p) e^{H_2(p)(T-t_0)}, \quad r \geq r_0. \quad (37)$$

- Thus, letting $r \rightarrow \infty$ in the last inequality we obtain that $P\{\tau_\infty \leq T\} = 0$, that is $P\{\tau_\infty > T\} = 1$.
- On the other hand, letting $r \rightarrow \infty$ in (36) implies

$$E \sup_{s \in [t_0 - \tau, T]} |x^n(s)|^p \leq H_1(p) e^{H_2(p)(T-t_0)}.$$

There exists a unique solution of Eq. (28) with bounded p -th moment, for any $p \geq 2$, which yields $\mathcal{A}_6$.

- Theorem 13 gives the rate of the closeness in the L^p sense

$$E \sup_{t \in [t_0 - \tau, T]} |x(t) - x^n(t)|^p \leq H \delta_n^p.$$

NEUTRAL STOCHASTIC DIFFERENTIAL EQUATION WITH CONSTANT TIME DELAY (DIFFERENTIAL FORM)

$$d[x(t) - D(t, x(t - \tau))] = f(t, x(t), x(t - \tau))dt + g(t, x(t), x(t - \tau))dw(t), \quad t \in [t_0, T], \quad (38)$$

with initial data $\eta = \{\eta(\theta), \theta \in [-\tau, 0]\}$.

- $\tau > 0 \leq t_0 < T$ are fixed real numbers
- $D : [t_0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $f : [t_0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $g : [t_0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d'}$ are Borel measurable
- η is $\mathcal{F}_{t_0}$ -measurable, V -valued random variable
- $x(t)$ is an $\mathbb{R}^d$ -valued random variable for any $t \in [t_0, T]$

In the work of Tan, Yuan (2019), an autonomous version of NSDDE (38) is observed:

$$\begin{aligned} d[x(t) - D(x(t - \tau))] &= f(x(t), x(t - \tau))dt \\ &\quad + g(x(t), x(t - \tau))dw(t), \quad t \geq 0 \end{aligned} \quad (39)$$

with initial data $\eta = \{\eta(\theta), \theta \in [-\tau, 0]\}$ which is $\mathbb{R}^d$ -valued, $\mathcal{F}_t$ -adapted process for which $\int_{-\tau}^0 |\eta(\theta)|^p d\theta < +\infty$ a.s. and $E \int_{-\tau}^0 |\eta(\theta)|^p d\theta < +\infty$ holds, $p \geq 2$, and where also:

- coefficients D, f and g are continuous;
- for every $i \in \{1, 2, 3\}$ there exist functions $V_i : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ for which there exist constants $L_i > 0$ and $\ell_i \geq 0$ such that for all $x, y \in \mathbb{R}^d$

$$0 \leq V_i(x, y) \leq L_i(1 + |x|^{\ell_i} + |y|^{\ell_i}).$$

- (S_1) Let there exist a positive constant K_1 such that for all $x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^d$

$$\langle x - D(y) - \tilde{x} + D(\tilde{y}), f(x, y) - f(\tilde{x}, \tilde{y}) \rangle \leq K_1 |x - \tilde{x}|^2 + |V_1(y, \tilde{y})|^2 |y - \tilde{y}|^2;$$

$$|f(x, y) - f(x, \tilde{y})| \leq V_1(y, \tilde{y}) |y - \tilde{y}|.$$

- (S_2) Let there exist a positive constant K_2 such that for all $x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^d$

$$|g(x, y) - g(\tilde{x}, \tilde{y})| \leq K_2 |x - \tilde{x}| + V_2(y, \tilde{y}) |y - \tilde{y}|.$$

- (S_3) Let $D(0) = 0$ and let for every $x, \tilde{x} \in \mathbb{R}^d$

$$|D(x) - D(\tilde{x})| \leq V_3(x, \tilde{x}) |x - \tilde{x}|.$$

LEMMA (TAN AND YUAN (2019))

Assumptions $\mathcal{S}_1 - \mathcal{S}_3$ imply that there exists a unique global solution x of Eq. (39) with initial data η such that for every $T > 0$ there exists a positive constant $C = C(T, \rho, \eta)$ such that

$$E \sup_{t \in [0, T]} |x(t)|^p \leq C.$$

Consider NSDE (38) of Itô type with initial data η in integral form:

NEUTRAL STOCHASTIC DIFFERENTIAL EQUATION WITH CONSTANT TIME DELAY (INTEGRAL FORM)

$$\begin{aligned} x(t) - D(t, x(t - \tau)) &= \eta(0) - D(t_0, \eta(-\tau)) \\ &+ \int_{t_0}^t f(s, x(s), x(s - \tau)) ds \\ &+ \int_{t_0}^t g(s, x(s), x(s - \tau)) dw(s), \quad t \in [t_0, T] \end{aligned} \quad (40)$$

Let us assume the following:

- $(\mathcal{A}_1)$ There exists a nonnegative constant r_0 such that

$$\|\eta\| \leq r_0 \text{ a.s.}$$

- $(\mathcal{A}_2)$ For all $t \in [t_0, T]$, neutral coefficient D satisfies $D(t, 0) = 0$ and it satisfies the polynomial condition: there exist nonnegative constants L_1 and ℓ_1 such that for every $t \in [t_0, T]$ and $y, \tilde{y} \in \mathbb{R}^d$,

$$|D(t, y) - D(t, \tilde{y})| \leq L_1(1 + |y|^{\ell_1} + |\tilde{y}|^{\ell_1})|y - \tilde{y}|.$$

- $(\mathcal{A}_3)$ There exist nonnegative constants $\bar{f}$ and $\bar{g}$ such that for all $t \in [t_0, T]$, coefficients f and g satisfy $f(t, 0, 0) \leq \bar{f}$ and $g(t, 0, 0) \leq \bar{g}$, and, also: there exist nonnegative constants $K_2, K_3, L_2, L_3, \ell_2$ and ℓ_3 such that $L_2 \vee L_3 > 0$ and that for every $t \in [t_0, T]$ and $x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^d$,

$$|f(t, x, y) - f(t, \tilde{x}, \tilde{y})| \leq K_2|x - \tilde{x}| + L_2(1 + |y|^{\ell_2} + |\tilde{y}|^{\ell_2})|y - \tilde{y}|,$$

$$|g(t, x, y) - g(t, \tilde{x}, \tilde{y})| \leq K_3|x - \tilde{x}| + L_3(1 + |y|^{\ell_3} + |\tilde{y}|^{\ell_3})|y - \tilde{y}|.$$

- On the basis of the Young inequality and consequences of the Hölder inequality, assumptions $\mathcal{A}_2$ and $\mathcal{A}_3$ imply that for every $t \in [t_0, T]$ and $x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^d$,

$$\begin{aligned} & \langle x - D(t, y) - \tilde{x} + D(t, \tilde{y}), f(t, x, y) - f(t, \tilde{x}, \tilde{y}) \rangle \\ & \leq (1 + K_2^2) |x - \tilde{x}|^2 + 9(L_1^2 + L_2^2)(1 + |y|^{\ell_1 \vee \ell_2} + |\tilde{y}|^{\ell_1 \vee \ell_2})^2 |y - \tilde{y}|^2, \end{aligned}$$

that is, S_1 is satisfied.

- Lemma (Tan, Yuan (2019)) guarantees that there exists a unique global solution $x = \{x(t), t \in [t_0 - \tau, T]\}$ of Eq. (40) with initial data η , which for $p \geq 2$ satisfies

$$E \sup_{t \in [t_0, T]} |x(t)|^p \leq C(E \|\eta\|^{\ell^j p}) \equiv C^{\ell^j p},$$

where

$$\ell = 1 + (\ell_1 \vee \ell_2 \vee \ell_3), \quad j = \left\lceil \frac{T - t_0}{\tau} \right\rceil.$$

- Let $T - t_0$ and τ have a rational ratio: there exist integers $n, n' > 0$, large enough, such that $\delta_n = (T - t_0)/n = \tau/n' < 1$.
- Let us define the equidistant partition of the time interval $[t_0 - \tau, T]$:

$$t_0 - \tau = t_{-n'} < \cdots < t_{-1} < t_0 < t_1 < \cdots < t_n = T. \quad (41)$$

Let us now impose the following assumption:

- $(\mathcal{B}_1)$ Function D allows Taylor expansion up to the order m_1 on the second argument and functions f and g allow Taylor expansions up to the orders m_2 and m_3 , respectively, on the third argument.

GOAL:

Approximation of the solution $x = \{x(t), t \in [t_0 - \tau, T]\}$ of Eq. (40) with initial condition η on partition (41) via stochastic processes noted as $x^n = \{x^n(t), t \in [t_0 - \tau, T]\}$, which are obtained by successive connecting of solutions of eqs.

$$\begin{aligned}
 x^n(t) &= \sum_{i=0}^{m_1} \frac{D_{(t, x^n(t_k - \tau))}^{(i)}}{i!} (x^n(t - \tau) - x^n(t_k - \tau))^i \\
 &= x^n(t_k) - D(t_k, x^n(t_k - \tau)) \\
 &\quad + \int_{t_k}^t \sum_{i=0}^{m_2} \frac{f_{(s, x^n(s), x^n(t_k - \tau))}^{(i)}}{i!} (x^n(s - \tau) - x^n(t_k - \tau))^i ds \\
 &\quad + \int_{t_k}^t \sum_{i=0}^{m_3} \frac{g_{(s, x^n(s), x^n(t_k - \tau))}^{(i)}}{i!} (x^n(s - \tau) - x^n(t_k - \tau))^i dw(s), \quad t \in [t_k, t_{k+1}],
 \end{aligned} \tag{42}$$

with the initial data η , $k = 0$ and $\{x^n(t_k + \theta), \theta \in [-\tau, 0]\}$, $k > 0$.

The stochastic process $x^n = \{x^n(t), t \in [t_0 - \tau, T]\}$ obtained this way is almost surely continuous process.

NOTATIONS:

For every $k \in \{0, \dots, n-1\}$ and $t \in [t_k, t_{k+1}]$, let:

$$\begin{aligned} \tilde{D}(t, x^n(t-\tau); x^n(t_k-\tau)) &= \sum_{i=0}^{m_1} \frac{D_{(t, x^n(t_k-\tau))}^{(i)}}{i!} (x^n(t-\tau) - x^n(t_k-\tau))^i; \\ F(t, x^n(t), x^n(t-\tau); x^n(t_k-\tau)) &= \int_{t_k}^t \sum_{i=0}^{m_2} \frac{f_{(s, x^n(s), x^n(t_k-\tau))}^{(i)}}{i!} (x^n(s-\tau) - x^n(t_k-\tau))^i ds; \\ G(t, x^n(t), x^n(t-\tau); x^n(t_k-\tau)) &= \int_{t_k}^t \sum_{i=0}^{m_3} \frac{g_{(s, x^n(s), x^n(t_k-\tau))}^{(i)}}{i!} (x^n(s-\tau) - x^n(t_k-\tau))^i dw(s). \end{aligned}$$

NOTATIONS:

For every $t \in [t_0, T]$ (for $\sum_{i \in \emptyset} a_i = 0$), let:

$$z^n(t - \tau) = \sum_{k=0}^{n-2} \chi_{[t_k, t_{k+1})}(t) x^n(t_k - \tau) + \chi_{[t_{n-1}, t_n]}(t) x^n(t_{n-1} - \tau);$$

$$y_t^n = x^n(t - \tau); \quad z_t^n = z^n(t - \tau).$$

THEN: Eqs. (42) can be written shortly as one equation:

$$x^n(t) - \tilde{D}(t, y_t^n; z_t^n) = \eta(0) - D(t_0, \eta(-\tau)) + \int_{t_0}^t F(s, x^n(s), y_s^n; z_s^n) ds$$

$$+ \int_{t_0}^t G(s, x^n(s), y_s^n; z_s^n) dw(s), \quad t \in [t_0, T]. \quad (43)$$

ASSUMPTIONS:

- (B_2) Functional $D^{(m_1+1)}$ satisfies following:

There exists a nonnegative constant $\bar{c}_1$ such that for all $t \in [t_0, T]$ it is $D_{(t,0)}^{(m_1+1)} \leq \bar{c}_1$ and $D^{(m_1+1)}$ satisfies the polynomial condition: there exist nonnegative constants L'_1 and ℓ'_1 such that for every $t \in [t_0, T]$ and $y, \tilde{y} \in \mathbb{R}^d$,

$$\|D_{(t,y)}^{(m_1+1)} - D_{(t,\tilde{y})}^{(m_1+1)}\|_{m_1+1} \leq L'_1(1 + |y|^{\ell'_1} + |\tilde{y}|^{\ell'_1})|y - \tilde{y}|.$$

- (B_3) Functionals $f^{(m_2+1)}$ and $g^{(m_3+1)}$ satisfy the polynomial condition on the third argument: there exist nonnegative constants L'_2, L'_3, ℓ'_2 and ℓ'_3 such that for every $t \in [t_0, T]$ and $x, y, \tilde{y} \in \mathbb{R}^d$,

$$\begin{aligned} \|f_{(t,x,y)}^{(m_2+1)} - f_{(t,x,\tilde{y})}^{(m_2+1)}\|_{m_2+1} &\leq L'_2(1 + |y|^{\ell'_2} + |\tilde{y}|^{\ell'_2})|y - \tilde{y}|; \\ \|g_{(t,x,y)}^{(m_3+1)} - g_{(t,x,\tilde{y})}^{(m_3+1)}\|_{m_3+1} &\leq L'_3(1 + |y|^{\ell'_3} + |\tilde{y}|^{\ell'_3})|y - \tilde{y}|. \end{aligned}$$

- $(\mathcal{B}_4)$ Functionals $f^{(m_2+1)}(\cdot, \cdot, 0)$ and $g^{(m_3+1)}(\cdot, \cdot, 0)$ are uniformly bounded: there exist nonnegative constants $\bar{f}'$ and $\bar{g}'$ such that

$$\sup_{(t,x) \in [t_0, T] \times \mathbb{R}^d} \|f_{(t,x,0)}^{(m_2+1)}\|_{m_2+1} \leq \bar{f}', \quad \sup_{(t,x) \in [t_0, T] \times \mathbb{R}^d} \|g_{(t,x,0)}^{(m_3+1)}\|_{m_3+1} \leq \bar{g}'.$$

THEOREM 1 (PART 1)

Assume that the conditions $\mathcal{A}_1 - \mathcal{A}_3$ and $\mathcal{B}_1 - \mathcal{B}_4$ are satisfied. Let the functional $D^{(i_1)}$, for $i_1 \in \{1, \dots, m_1\}$, considered as a function defined on $[t_0, T] \times \mathbb{R}^d$, and the functionals $f^{(i_2)}$ and $g^{(i_3)}$, for $i_2 \in \{1, \dots, m_2\}$ and $i_3 \in \{1, \dots, m_3\}$, considered as functions defined on $[t_0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, satisfy following local Lipschitz conditions:

$$(\forall k \in \mathbb{N})(\exists K_k \geq 0)(\forall t \in [t_0, T])(\forall \alpha \in \{1, 2, 3\})(\forall i_\alpha \in \{1, \dots, m_\alpha\})$$

$$(\forall x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^d : |x| \vee |y| \vee |\tilde{x}| \vee |\tilde{y}| \leq k)$$

$$\|D_{(t,y)}^{(i_1)} - D_{(t,\tilde{y})}^{(i_1)}\|_{i_1} \leq K_k |y - \tilde{y}|;$$

$$\|f_{(t,x,y)}^{(i_2)} - f_{(t,\tilde{x},\tilde{y})}^{(i_2)}\|_{i_2} \leq K_k (|x - \tilde{x}| + |y - \tilde{y}|);$$

$$\|g_{(t,x,y)}^{(i_3)} - g_{(t,\tilde{x},\tilde{y})}^{(i_3)}\|_{i_3} \leq K_k (|x - \tilde{x}| + |y - \tilde{y}|).$$

THEOREM 1 (PART 2)

Then, Eq. (43) has a unique global solution

$x^n = \{x^n(t), t \in [t_0 - \tau, T]\}$ with initial data η which, for every $p > 0$ satisfies

$$E \sup_{s \in [t_0, T]} |x^n(s)|^p \leq C_j^{(\gamma^{j-1}p)} (E \|\eta\|^{\gamma^j p}) \equiv C_j^{(\gamma^{j-1}p)},$$

where

$$\gamma = 1 + \max \{ \ell_i, \ell'_i + m_i + 1 \mid i \in \{1, 2, 3\} \}, \quad j = \left\lceil \frac{T - t_0}{\tau} \right\rceil.$$

Let $L_r = \{x \in \mathbb{R}^d \mid |x| \leq r\}$.

One can consider slightly different conditions and obtain the same existence and uniqueness result.

NOTATIONS:

$$\begin{aligned} A_1 &= |D(t, \phi)| \vee \|D_{(t, \phi)}^{(m_1+1)}\|_{j_1+1}, & A_2 &= \|D_{(t, \phi)}^{(i_1)}\|_{i_1}, \\ B_1 &= |f(t, 0, \phi)| \vee \|f_{(t, 0, \phi)}^{(m_2+1)}\|_{j_2+1}, & B_2 &= \|f_{(t, 0, \phi)}^{(i_2)}\|_{i_2}, \\ C_1 &= |g(t, 0, \phi)| \vee \|g_{(t, 0, \phi)}^{(m_3+1)}\|_{j_3+1}, & C_2 &= \|g_{(t, 0, \phi)}^{(i_3)}\|_{i_3}. \end{aligned}$$

ASSUMPTIONS ($i, j, k \in \{1, 2\}$):

- $(C_{i,j,k})$ There exists a nondecreasing real function $h : [r_0, +\infty) \rightarrow \mathbb{R}_+$ such that for every $r \geq r_0$ (and $i_1 \in \{0, \dots, m_1\}$) (and $i_2 \in \{0, \dots, m_2\}$) (and $i_3 \in \{0, \dots, m_3\}$),

$$\sup_{(t, \phi) \in [0, T] \times_r L} (A_i \vee B_j \vee C_k) \leq h(r) \text{ a.s.}$$

- $(\mathcal{D}_1)$ For any $r \geq 0$ there exists $K_r \geq 0$ such that for all $t \in [t_0, T]$ and $x, \tilde{x}, y \in L_r$

$$\begin{aligned} \|f_{(t,x,y)}^{(i_2)} - f_{(t,\tilde{x},y)}^{(i_2)}\|_{i_2} &\leq K_r |x - \tilde{x}|, \quad i_2 = \overline{0, j_2}; \\ \|g_{(t,x,y)}^{(i_3)} - g_{(t,\tilde{x},y)}^{(i_3)}\|_{i_3} &\leq K_r |x - \tilde{x}|, \quad i_3 = \overline{0, j_3}. \end{aligned}$$

THEOREM 2

Under the assumptions of $\mathcal{A}_1$, $\mathcal{B}_1$, $\mathcal{D}_1$ and one of the assumptions $\mathcal{C}_{i,j,k}$, $i, j, k \in \{1, 2\}$, NSDDE (43) with initial condition η has a unique global solution $x^n = \{x^n(t), t \in [t_0 - \tau, T]\}$.

FURTHER ASSUMPTIONS:

- (C_1) There exist nonnegative constants $K_2'', L_2'', K_3'', L_3'', \ell_2''$ and ℓ_3'' such that for every $t \in [t_0, T]$ and $x, \tilde{x}, y, \tilde{y} \in \mathbb{R}^d$,

$$\|f_{(t,x,y)}^{(m_2+1)} - f_{(t,\tilde{x},\tilde{y})}^{(m_2+1)}\|_{m_2+1} \leq K_2''|x - \tilde{x}| + L_2''(1 + |y|^{\ell_2''} + |\tilde{y}|^{\ell_2''})|y - \tilde{y}|;$$

$$\|g_{(t,x,y)}^{(m_3+1)} - g_{(t,\tilde{x},\tilde{y})}^{(m_3+1)}\|_{m_3+1} \leq K_3''|x - \tilde{x}| + L_3''(1 + |y|^{\ell_3''} + |\tilde{y}|^{\ell_3''})|y - \tilde{y}|.$$

- (C_2) Functional D satisfies the polynomial condition on the first (time) argument:

There exist nonnegative constants L_0 and ℓ_0 such that for every $s_1, s_2 \in [t_0, T]$ and $y \in \mathbb{R}^d$,

$$|D(s_1, y) - D(s_2, y)| \leq L_0(1 + s_1^{\ell_0} + s_2^{\ell_0})|s_1 - s_2|.$$

- (C_3) Initial data η satisfies the polynomial condition:
 There exist nonnegative constants Λ and λ such that for every $\theta', \theta'' \in [-\tau, 0]$,

$$|\eta(\theta') - \eta(\theta'')| \leq \Lambda(1 + |\theta'|^\lambda + |\theta''|^\lambda)|\theta' - \theta''|.$$

PROPOSITION 1

Let x^n be a solution of Eq. (43) with initial data η . Assume that the conditions $\mathcal{A}_1 - \mathcal{A}_3$, $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_4$ and $\mathcal{C}_1 - \mathcal{C}_3$ are satisfied. Then, for every $r > 0$, following holds

$$E \sup_{s \in [t_k, t]} |x^n(s) - x^n(t_k)|^r \leq \Theta_j^{(\gamma^{j-1}r)} \delta_n^{r/2},$$

where $\Theta_j^{(\gamma^{j-1}r)}$ is a constant which is independent of n and δ_n .

THEOREM 2

Let all the assumptions from the previous Proposition 1 hold and let x be the unique solution of Eq. (40) with initial data η . Then, for all $\bar{p} \geq 0$ and $q > 1$,

$$E \sup_{t \in [t_0 - \tau, T]} |x(t) - x^n(t)|^{\bar{p}} \leq \bar{C} \delta_n^{(m+1)\bar{p}/2 - 1/q},$$

where $m = m_1 \wedge m_2 \wedge m_3$ and $\bar{C}$ is a constant which is independent of n and δ_n .

THEOREM 3

Let all the assumptions from the previous Theorem 2. Then $(x^n)_{n \in \mathbb{N}}$ converges to x almost surely.

EXAMPLE: Autonomous NSDE

$$\begin{aligned}
 x(t) - D(t, x(t - \tau)) = & \eta(0) - D(t_0, \eta(-\tau)) \\
 & + \int_{t_0}^t f(s, x(s), x(s - \tau)) ds \\
 & + \int_{t_0}^t g(s, x(s), x(s - \tau)) dw(s), \quad t \in [t_0, T],
 \end{aligned} \tag{44}$$

with initial condition $\eta(\theta) = 0$ for each $\theta \in [-\tau, 0]$.

- One-dim. ($d' = 1$) Wiener process w , Eq. (44) is d -dimensional

- $T > t_0 \geq 0$ and $\tau > 0$ are such that $(T - t_0)/\tau$ is rational

For $e = (1, \dots, 1) \in \mathbb{R}^d$, the functionals $D : [t_0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $f, g : [t_0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ are defined as

$$D(t, y) = |y|^4 y, \quad f(t, x, y) = x + |y|^2 y, \quad g(t, x, y) = x + |y|^2 e.$$

All of the previous assumptions are satisfied with

$m_1 = m_2 = m_3 = 2$, so, Theorem 2 provides the L^p -estimate

$$E \sup_{t \in [t_0 - \tau, T]} |x(t) - x^n(t)|^{\bar{p}} \leq \bar{C} \delta_n^{3\bar{p}/2 - 1/q}.$$

CONCLUSION

- The goal was to construct the approximate solutions of various types of SDEs defined on a partition of the time interval.
- The coefficients of initial equation are approximated by their Taylor series in the case when they behave like polynomials.
- The closeness of the initial and approximate solutions is estimated in the sense of the L^p -norm and with probability one.
- Examples which illustrate our findings are presented.

THANK YOU FOR YOUR ATTENTION!