

Reflected diffusions in generalised parabolic domains

Aleksandar Mijatović
Department of Statistics



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Joint with Miha Brešar, Juan Pablo Chavez Ochoa, Mikhail Menshikov,
Isao Sauzedde and Andrew Wade

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What happens with Brownian particle B_t in $\mathbb{R}^d$ as $t \uparrow \infty$?

Classical result:

$\mathbf{d} \geq 3 \implies$ Transience: $\|B_t\|_d \rightarrow \infty$ as $t \rightarrow \infty$ a.s.

$\mathbf{d} = 1, 2 \implies$ Recurrence: $\exists r_0 > 0$ s.t. $\liminf_{t \rightarrow \infty} \|B_t\|_d < r_0$ a.s.

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Reason: Harmonic function $u : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}$, $\Delta u = 0$, equals

$d \geq 3 \implies u(x) = 1/\|x\|_d^{d-2}$

$d = 2 \implies u(x) = \log \|x\|_2$

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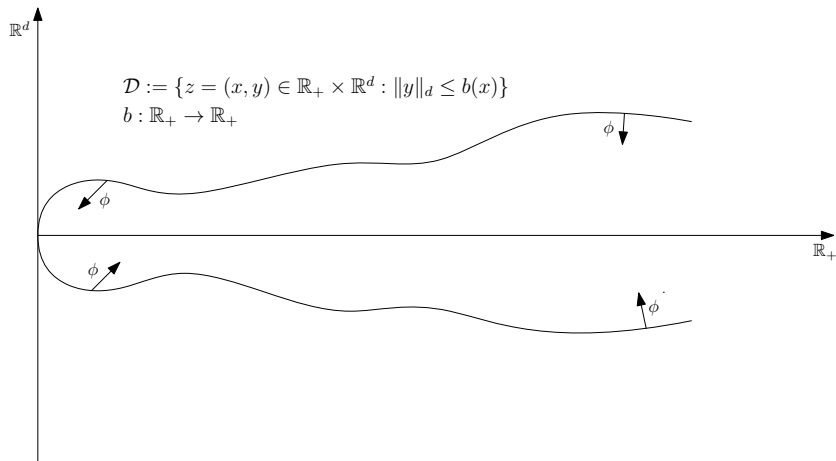
Hence

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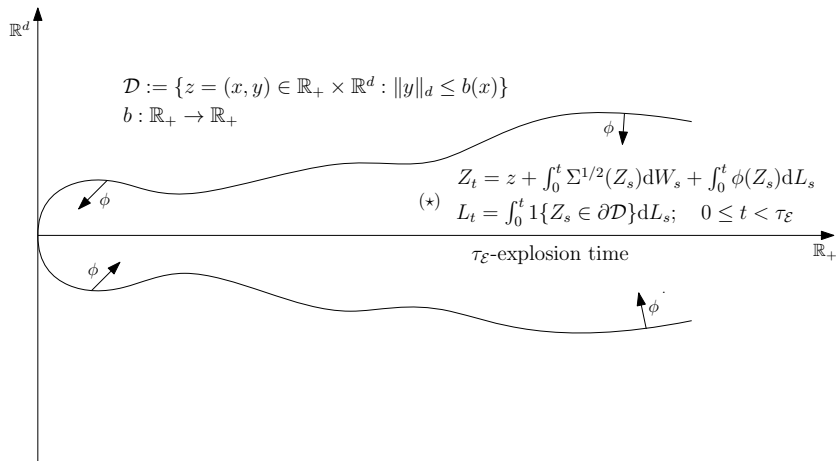
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Question: can spacial restriction of B_t change its behaviour as $t \uparrow \infty$?

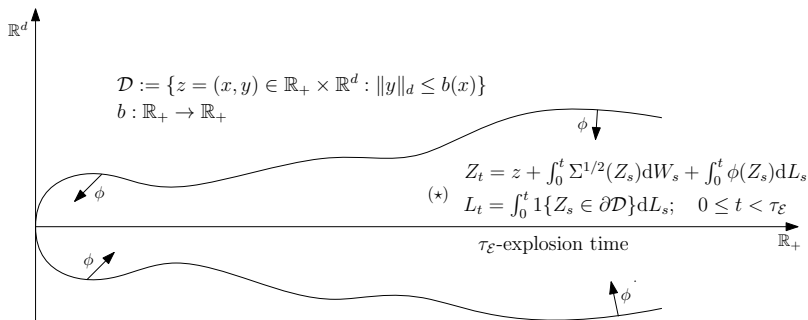
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Assumptions:

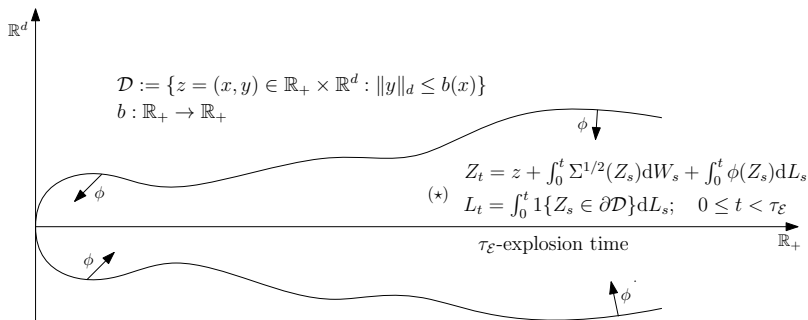
(D1) $\mathcal{D}$ is a smooth domain.

(C1) Σ is bounded, Lipschitz and uniformly elliptic: $\langle \Sigma(z)v, v \rangle \geq \delta \|v\|_{d+1}^2$.

(V1) $\phi : \partial \mathcal{D} \rightarrow \mathbb{R}^{d+1}$ C^2 vector field, $\sup_{z \in \partial \mathcal{D}} \|\phi(z)\|_{d+1} < \infty$ and $\inf_{z \in \partial \mathcal{D}} \langle \phi(z), n(z) \rangle > 0$.

Here $n(z)$ is the inwards pointing normal for $z \in \partial \mathcal{D}$.

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Generator $\mathcal{L}f(z) = \lim_{t \rightarrow 0} \mathbb{E}_z[f(Z_t) - f(z)]/t$ of Z on $f \in C^2(\mathcal{D})$ equals

$$\mathcal{L}f = \frac{1}{2} \text{Tr}(\Sigma \nabla \otimes \nabla(f)) \text{ with core } \{h \in C_b^2(\mathcal{D}) : \text{for } z \in \partial \mathcal{D}, \langle \nabla h(z), \phi(z) \rangle = 0\}$$

Existence and uniqueness of (Z, L)

$$(\star) \quad Z_t = z + \int_0^t \Sigma^{1/2}(Z_s) dW_s + \int_0^t \phi(Z_s) dL_s \quad \& \quad L_t = \int_0^t \mathbb{1}\{Z_s \in \partial\mathcal{D}\} dL_s^Z$$

Theorem (MMW23)

Let (D1), (C1), (V1) hold on $\mathcal{D} = \{z = (x, y) : \|y\|_d \leq b(x)\}$. Then there exists a strong solution $(Z, L, \tau_{\mathcal{E}})$ satisfying $(\star)$ and pathwise uniqueness holds. In particular, $(\star)$ defines a continuous strong Markov process Z on the time interval $[0, \tau_{\mathcal{E}})$ and

$$\lim_{t \uparrow \tau_{\mathcal{E}}} \|Z_t\|_{d+1} = \lim_{t \uparrow \tau_{\mathcal{E}}} L_t = \infty, \text{ on } \{\tau_{\mathcal{E}} < \infty\}.$$

Proof via approximation by increasing sequence of bounded domains and applying existence/uniqueness [LS84]. Main technical obstacle is verifying that explosive paths do not oscillate:

$$\liminf_{t \uparrow \tau_{\mathcal{E}}} \|Z_t\|_{d+1} = \infty \quad \text{on } \{\tau_{\mathcal{E}} < \infty\}.$$

Structure of the talk

As $x \rightarrow \infty$, boundary $b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is **sub-linear** ($b'(x) \rightarrow 0$) and

- 1 Asymptotically **oblique** reflection “away from” the origin $\rightarrow$:

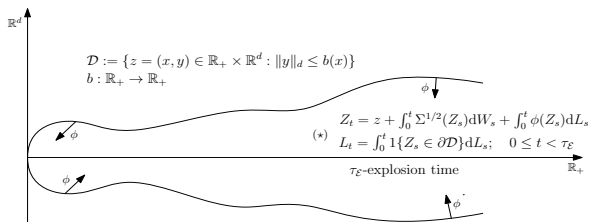
$$\langle \phi(z), e_x \rangle \rightarrow s_0 > 0$$

- 2 Asymptotically **normal** reflection:

$$\langle \phi(z), e_x \rangle \rightarrow 0$$

- 3 Asymptotically **oblique** reflection “towards” the origin $\leftarrow$:

$$\langle \phi(z), e_x \rangle \rightarrow s_0 < 0$$



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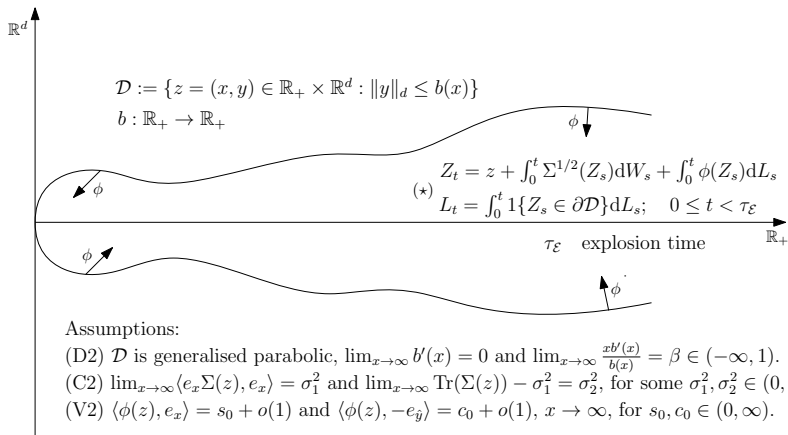
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Think $\mathbf{b}(\mathbf{x}) = \mathbf{a}x^\beta$ for large $x \gg 1$, some $a > 0$ and $\beta \in (-\infty, 1)$.

Explosion and strong law

Last piece of notation: for $x \in \mathbb{R}_+$,

$$B(x) := \int_0^x b(u)du, \text{ and } B(\infty) = \lim_{x \rightarrow \infty} B(x) \in [0, \infty].$$

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Suppose that (D1), (D2), (C), (V), and (A) hold.

Explosion: If $B(\infty) < \infty$, then $\sup_{z \in \mathcal{D}} \mathbb{E}_z \tau_{\mathcal{E}} < \infty$, and

$$\lim_{t \uparrow \tau_{\mathcal{E}}} X_t = \lim_{t \uparrow \tau_{\mathcal{E}}} L_t = \infty, \mathbb{P}_z\text{-a.s. for every } z \in \mathcal{D}.$$

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Non-explosion: If $B(\infty) = \infty$, then, for every $z \in \mathcal{D}$, $\mathbb{P}_z(\tau_{\mathcal{E}} = \infty) = 1$, and we have the **strong law**

$$\lim_{t \rightarrow \infty} \frac{B(X_t)}{t} = \lim_{t \rightarrow \infty} \frac{B(\|Z_t\|_{d+1})}{t} = \frac{s_0 \sigma^2}{2c_0}, \mathbb{P}_z\text{-a.s.}$$

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- If $\beta \in (-1, 1)$, then $\mathbb{P}_z(\tau_{\mathcal{E}} = \infty) = 1$, and, $\mathbb{P}_z\text{-a.s.}$,

$$\lim_{t \rightarrow \infty} t^{-\frac{1}{1+\beta}} \|Z_t\|_{d+1} = \left(\frac{(1+\beta)s_0 \sigma^2}{2ac_0} \right)^{\frac{1}{1+\beta}} \quad (\text{superdiffusive transience})$$

Explosion and strong law - Remarks

- For any $\beta < 1$, we have $\|Z_t\|_{d+1}/\sqrt{t} \rightarrow \infty$, a.s.; this **superdiffusive transience** is an example of the phenomenon of **anomalous diffusion**.
- Around critical window $\beta = -1$, non-explosion with **arbitrarily fast** escape; e.g. $b(x) = 1/(x \log x)$ yields **double exponential** growth.

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- Explosion to infinity driven by reflection has not been documented, but cf. **corner trapping** in domains with non-smooth boundaries (VARADHAN & WILLIAMS, 1985) and explosions for diffusions on **manifolds** (GRIGOR'YAN, 1999).
- **Existence** results for diffusions with oblique reflections [LS84, SV71] do not admit explosion (not applicable to unbounded, narrowing domain).

CLT in the superdiffusive regime

Theorem (MSW26)

Assume $\beta \in (-1/3, 1)$ and $b(x) = a_\infty x^\beta + o(x^{(3\beta-1)/2})$. Recall

$$X_t / t^{\frac{1}{1+\beta}} \xrightarrow{t \rightarrow \infty} c_1 := \left(\frac{(1+\beta)s_0\sigma^2}{2a_\infty c_0} \right)^{\frac{1}{1+\beta}} \quad a.s.$$

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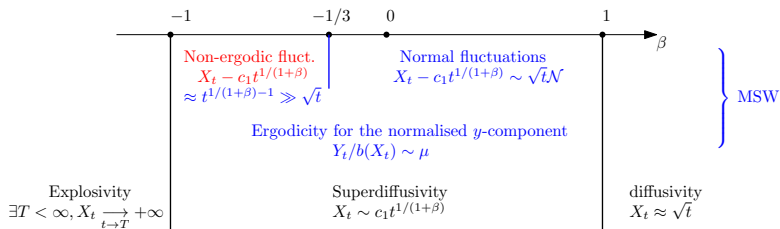
$$\left(\frac{X_t - c_1 t^{\frac{1}{1+\beta}}}{\sqrt{t}}, \frac{Y_t}{a_\infty c_1^\beta t^{\frac{\beta}{1+\beta}}} \right) \longrightarrow \mathcal{N} \otimes \mu, \quad \text{as } t \rightarrow \infty,$$

where $\mathcal{N}$ is a centred Gaussian law on $\mathbb{R}$ with variance v given by

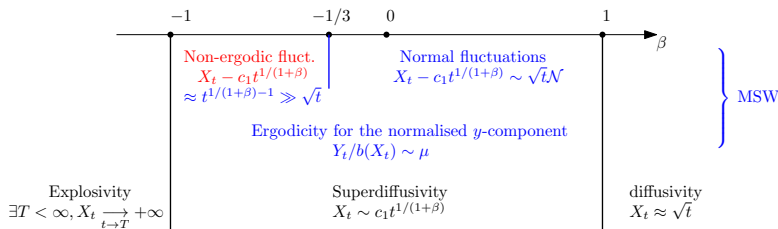
$$\begin{aligned} v := & c_1^{2-2\beta} \frac{1+\beta}{1+3\beta} \left(\Sigma_{0,0}^\infty + \frac{s_0^2}{c_0^2} a_\infty \left(\sum_{j=1}^d \Sigma_{0,j}^\infty \int_{\mathbb{B}^d} y_j \mu(dy) \right. \right. \\ & \left. \left. + a_\infty \sum_{j,k=1}^d \Sigma_{j,k}^\infty \int_{\mathbb{B}^d} y_j y_k \mu(dy) \right) \right) \end{aligned}$$

and μ a stationary measure of a reflected diffusion on $\mathbb{B}^d$.

CLT in the superdiffusive regime



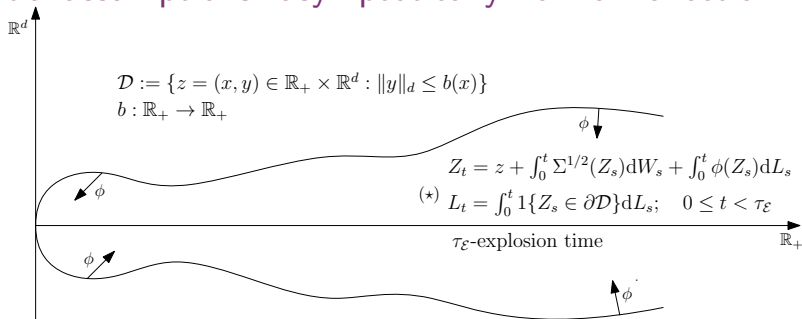
CLT in the superdiffusive regime



Loss of CLT at $\beta = -1/3$: we prove $(X_t - c_1 t^{\frac{1}{1+\beta}})/\sqrt{t}$ **does not** converge weakly for $\beta \in (-1, -1/3)$.

Non-ergodic fluctuations appear to yield an almost sure limit for domains that shrink faster than $b(x) \propto x^{-1/3}$

Model assumptions: asymptotically normal reflection



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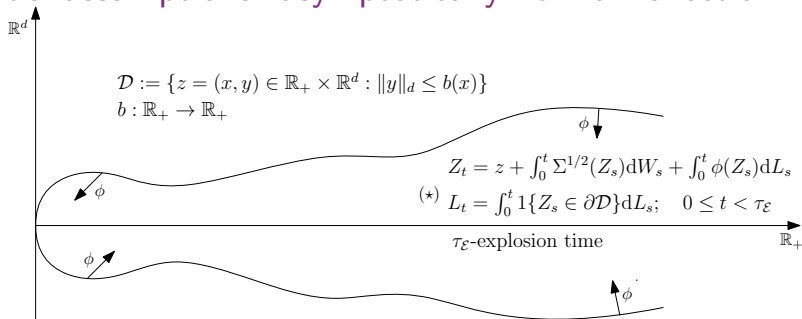
(D2) $\mathcal{D}$ is generalised parabolic, $\lim_{x \rightarrow \infty} b'(x) = 0$ and $\lim_{x \rightarrow \infty} \frac{xb'(x)}{b(x)} = \beta \in (-\infty, 1)$.

(C2) $\lim_{x \rightarrow \infty} \langle e_x \Sigma(z), e_x \rangle = \sigma_1^2$ and $\lim_{x \rightarrow \infty} \text{Tr}(\Sigma(z)) - \sigma_1^2 = \sigma_2^2$, for some $\sigma_1^2, \sigma_2^2 \in (0, \infty)$.

(V2) $\langle \phi(z), e_x \rangle = b'(x)s_0 + o(1)$ and $\langle \phi(z), -e_{\hat{y}} \rangle = c_0 + o(1)$, $x \rightarrow \infty$, for $s_0, c_0 \in (0, \infty)$.

Think $\mathbf{b}(\mathbf{x}) = \mathbf{a}x^\beta$ for large $x \gg 1$, some $a > 0$ and $\beta \in (-\infty, 1)$ and $\phi(\mathbf{z}) = \mathbf{n}(\mathbf{z}) = \frac{b'(x)\mathbf{e}_x - \mathbf{e}_{\hat{y}}}{\sqrt{b'(x)^2 + 1}}$, $\mathbf{z} \in \partial\mathcal{D}$, $\mathbf{e}_{\hat{y}} = (0, \frac{\mathbf{y}}{\|\mathbf{y}\|_d})$, $\mathbf{e}_x = (1, 0) \in \mathbb{R} \times \mathbb{R}^d$.

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[BMW25](Thm 4.1): $\mathbb{P}_z(\tau_{\mathcal{E}} = \infty) = 1$ for all $z \in \mathcal{D}$ and Z is diffusive:

$$\sup_{t \in \mathbb{R}_+} \mathbb{E}_z \|Z_t\|_{d+1}^2 / (1+t) < \infty.$$

Phase transition: recurrence/transience

Critical exponent for the phase transitions will be $\beta_c = \frac{c_0 \sigma_1^2}{s_0 \sigma_2^2}$

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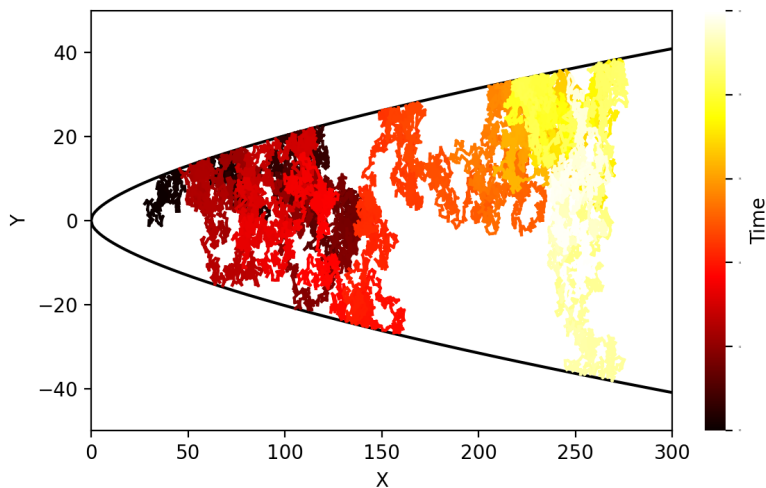
Theorem (BMW25)

Suppose that Assumptions (D2), (C2) and (V2) hold and the process Z solves SDE $(\star)$. Then the following statements hold for all starting points $z \in \mathcal{D}$:

- (a) if $\beta \in (-\infty, \beta_c)$, then Z is recurrent;*
- (b) if $\beta \in (\beta_c, 1)$, then Z is transient;*
- (c) if $\beta = \beta_c$ and additional assumptions are satisfied, then Z is recurrent.*

This theorem extends [Pin09] where $(d+1)$ -dimensional BM with normal reflection was studied for $\beta \geq 0$. In that case $\beta_c = 1/d$.

Transience: $\beta = 0.65 > 0.5 = \beta_c$



Martingale in interior, pushed away at boundary. Normal reflection, $\sigma_1^2 = 0.5$, $\sigma_2^2 = 1$. (If $\sigma_1^2 = \sigma_2^2 = 1$ in $\mathcal{D} \subset \mathbb{R}^2$, Z not transient.)

Return times of the reflected process

Tails of return times of $Z = (X, Y) \in \mathbb{R}_+ \times \mathbb{R}^d$ below any level $r > 0$,

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Proof uses ideas from [BM26], more on this perhaps later ...

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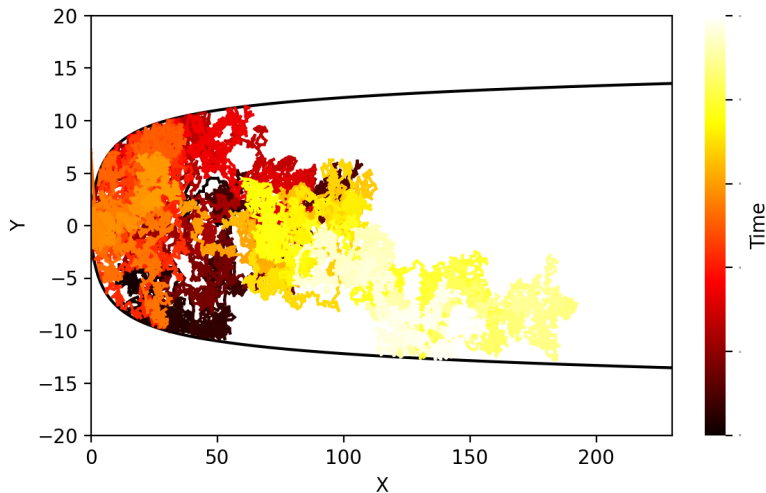
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- (b) $\beta \in (-\infty, \beta_c)$: *for any $\varepsilon > 0$, level $r > 0$ and starting point $z \in \mathcal{D} \cap (r, \infty) \times \mathbb{R}^d$, there exist $C_1, C_2 \in (0, \infty)$ such that*

$$C_1 t^{-m_c - \varepsilon} \leq \mathbb{P}_z(\varsigma_r \geq t) \leq C_2 t^{-m_c + \varepsilon} \quad \text{for all } t \in (1, \infty).$$

Proof uses ideas from [BM26], more on this perhaps later ...

Null recurrence: $\beta = 0.1 < 1 = \beta_c$ ($m_c = 0.45$)



Normally reflected BM in an unbounded expanding domain. Tail of return time decays as $t^{-0.45}$, $t \rightarrow \infty$. ("less" recurrent than scalar BM.)

Invariant distribution and the rate of convergence

Theorem (BMW25)

Let (D2), (C2), (V2) hold & $\beta < -\beta_c$. Then reflected process Z in $(\star)$, has a unique invariant distribution π . Define

$M_c := -(1 + \beta/\beta_c)/2 > 0$ and pick $\epsilon > 0$.

(a) $\exists$ constants $c_\pi, C_\pi \in (0, \infty)$ such that, for $r \in [1, \infty)$,

$$c_\pi r^{-2M_c - \epsilon} \leq \pi(\{z \in \mathcal{D} : \|z\|_{d+1} \geq r\}) \leq C_\pi r^{-2M_c + \epsilon};$$

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(b) for any starting point $z \in \mathcal{D}$ of Z , $\exists$ constants $c_{TV}, C_{TV} \in (0, \infty)$ such that, for $t \in [1, \infty)$,

$$c_{TV} t^{-M_c - \epsilon} \leq \|\mathbb{P}_z(Z_t \in \cdot) - \pi\|_{TV} \leq C_{TV} t^{-M_c + \epsilon}.$$

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Let (D2), (C2), (V2) hold & $\beta < -\beta_c$. Then reflected process Z in $(\star)$, has a unique invariant distribution π . Define

$M_c := -(1 + \beta/\beta_c)/2 > 0$ and pick $\epsilon > 0$.

(a) $\exists$ constants $c_\pi, C_\pi \in (0, \infty)$ such that, for $r \in [1, \infty)$,

$$c_\pi r^{-2M_c - \epsilon} \leq \pi(\{z \in \mathcal{D} : \|z\|_{d+1} \geq r\}) \leq C_\pi r^{-2M_c + \epsilon};$$

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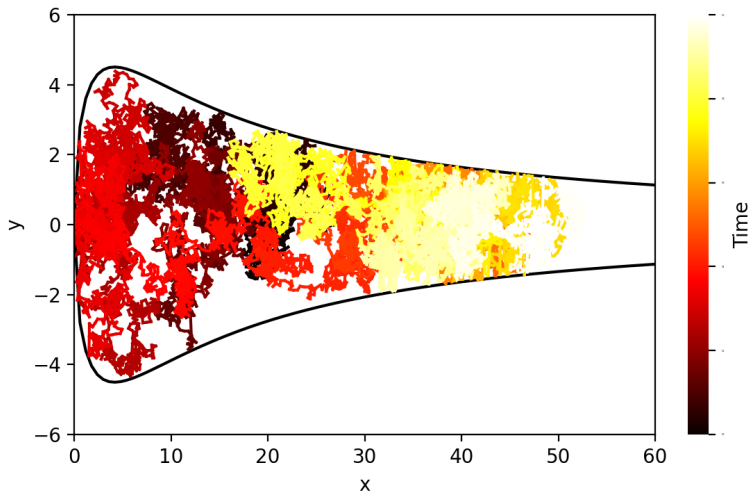
In particular, as a corollary, we get $\pi(\partial\mathcal{D}) = 0$ and

$$\lim_{t \rightarrow \infty} \frac{\log \|\mathbb{P}_z(Z_t \in \cdot) - \pi\|_{TV}}{\log t} = \frac{1}{2} \lim_{r \rightarrow \infty} \frac{\log \pi(\{z \in \mathcal{D} : \|z\|_{d+1} > r\})}{\log r} = -M_c.$$

Proof based on lower bounds from [BM26]: pathwise arguments yield

(a), while (b) requires idea from [Hai08].

Positive recurrence: $\beta = -1.2 < -1 = -\beta_c$ ($M_c = 0.1$)



Normally reflected Brownian motion in an unbounded narrowing domain.
Tail decay: $\pi(\|z\|_2 > r) \approx r^{-0.2}$; Convergence in TV: $t^{-0.1}$ as $t \rightarrow \infty$.

Remarks

Heuristic dynamics [Pin09]: $dL_t \approx \sigma_2^2/(2c_0b(x))dt$, when X_t is at level x .

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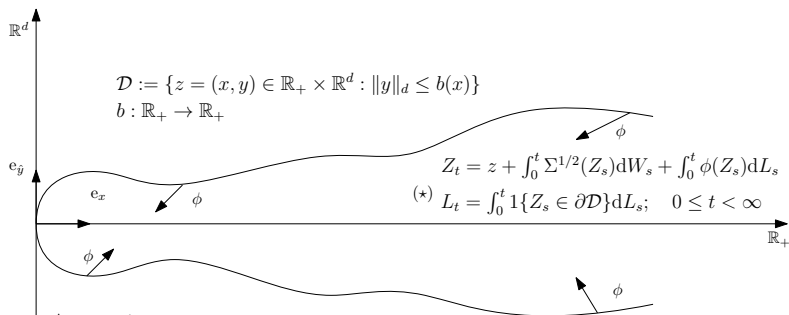
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Lyapunov function have been applied to reflecting BM **with interior drift** to get **upper bounds** on tails of invariant distribution and TV rate [Sar17].

Novelty of our approach:

- 1 Normally reflected BM exhibits behaviours from **transience to stability** in **any dimension** with **no interior drift**.
- 2 **Single class** of Lyapunov functions covers all non-critical cases.
- 3 **Lower bounds** on the **tail** of invariant distribution and on the **rate** of convergence, that **match upper bounds**.

Asymptotically oblique reflection “towards” the origin ←



$$\mathcal{D} := \{z = (x, y) \in \mathbb{R}_+ \times \mathbb{R}^d : \|y\|_d \leq b(x)\}$$

$$b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$$

$$Z_t = z + \int_0^t \Sigma^{1/2}(Z_s) dW_s + \int_0^t \phi(Z_s) dL_s$$

$$(*) \quad L_t = \int_0^t 1\{Z_s \in \partial\mathcal{D}\} dL_s; \quad 0 \leq t < \infty$$

Assumptions:

(D) $\mathcal{D}$ is generalised parabolic, $\lim_{x \rightarrow \infty} b'(x) = b''(x) = 0$ and $\lim_{x \rightarrow \infty} \frac{xb'(x)}{b(x)} = \beta \in (-\infty, 1)$.

(C) $\Sigma = \sigma\sigma^T$ is uniformly elliptic on $\mathcal{D}$ and $\sup_{z \in \mathcal{D}} \|\Sigma(z)\| < \infty$.

(V) $\sup_{z \in \partial\mathcal{D}} \|\phi(z)\|_{d+1} < \infty$ and reflection points asymptotically towards the origin:

$$\exists \delta > 0: \langle \phi(z), e_x \rangle < -\delta + o(1) \text{ and } \langle \phi(z), e_{\hat{y}} \rangle > \delta + o(1) \text{ as } x \rightarrow \infty.$$

Think $b(x) = ax^\beta$ for large $x \gg 1$, some $a > 0$ and $\beta \in (-\infty, 1)$.

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Tail decay of the stationary distribution π

Define
$$I_b(r) := \int_1^r \frac{du}{b(u)} \quad \text{for all } r \in [1, \infty)$$

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Theorem (BCM26: invariant distribution π)

Let (D) , (C) , (V) hold with $\beta < 1$. Then Z has invariant distribution π on $\mathcal{D}$ and $\exists C_\pi^-, c_\pi^-, c_\pi^+, C_\pi^+ \in (0, \infty)$ s.t., $\forall r \geq 1$,

$$C_\pi^- \exp(-c_\pi^- I_b(r)) \leq \pi(\|\cdot\|_{d+1} \geq r) \leq C_\pi^+ \exp(-c_\pi^+ I_b(r)).$$

Lower bounds follow from [BM26] for $\beta \in (-1, 1)$.

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Thus, $\forall \varepsilon > 0$ exist $C_{\pi,\varepsilon}^-, c_{\pi,\varepsilon}^-, c_{\pi,\varepsilon}^+, C_{\pi,\varepsilon}^+ \in (0, \infty)$ s.t., $\forall r \geq 1$,

$$C_{\pi,\varepsilon}^- \exp(-c_{\pi,\varepsilon}^- r^{1-\beta+\varepsilon}) \leq \pi(\|\cdot\|_{d+1} \geq r) \leq C_{\pi,\varepsilon}^+ \exp(-c_{\pi,\varepsilon}^+ r^{1-\beta-\varepsilon}).$$

If $\exists C, c > 0$ s.t. $cr^\beta \leq b(r) \leq Cr^\beta$ as $r \rightarrow \infty$, we can take $\varepsilon = 0$.

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Theorem (BCM26: rates of convergence to stationarity)

Let (D) , (C) , (V) hold with $\beta < 1$ and recall $B(x) := \int_0^x b(u)du$.

(a) $\beta \in [0, 1)$: $\forall z \in \mathcal{D}$ and $\varepsilon > 0$, $\exists C_L, c_L, c_U, C_U \in (0, \infty)$ s.t.

$$C_L \exp(-c_L t^{\frac{1-\beta}{1+\beta} + \varepsilon}) \leq \|\mathbb{P}_z(Z_t \in \cdot) - \pi\|_{TV} \leq C_U \exp(-c_U t^{\frac{1-\beta}{1+\beta} - \varepsilon}),$$

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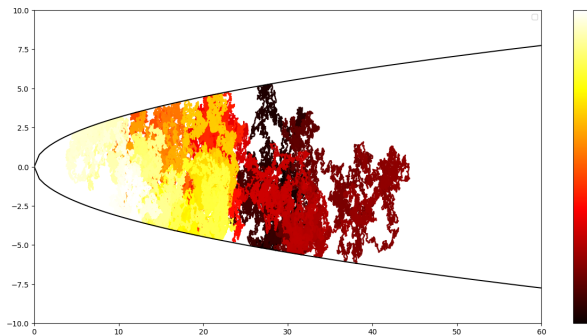
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(c) $B(\infty) < \infty \iff Z$ is uniformly ergodic: $\exists c, C \in (0, \infty)$ s.t.

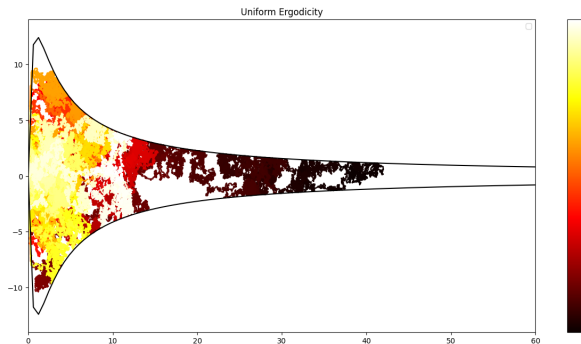
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Oblique reflection $\leftarrow$: subexponential ergodicity $\beta \in (0, 1)$



Subexponential ergodicity with $\beta = 0.5$

Oblique reflection $\leftarrow$: uniform ergodicity $\beta < -1$



Uniform ergodicity with $\beta = -1.1$

Oblique $\leftarrow$: **descent from infinity** under uniform erg.

Recall $B(x) = \int_0^x b(u)du$ and SDE satisfied by (Z^z, L^z) , for any $z \in \mathcal{D}$:

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Let (D) , (C) , (V) hold and $B(\infty) < \infty$. There exists a process $(\mathcal{Z}_t, \mathcal{L}_t)_{t \geq 0}$ under $\mathbb{P}_\infty$ such that, for every $\delta > 0$, we have

$$(\mathcal{Z}_{\delta+t}^z, L_{\delta+t}^z - L_\delta^z)_{t \geq 0} \xrightarrow{d} (\mathcal{Z}_{\delta+t}, \mathcal{L}_{\delta+t} - \mathcal{L}_\delta)_{t \geq 0} \quad \text{as } \|z\|_{d+1} \rightarrow \infty.$$

Remark. Limit $(\mathcal{Z}_t, \mathcal{L}_t)_{t \geq 0}$ is **not unique**. Limits are parametrised by $t_0 > 0$ such that $\mathcal{L}_{t_0} = 0$. But law of $\mathcal{Z}$ is **unique**! 

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Recall $B(x) = \int_0^x b(u)du$, let $B(\infty) < \infty$, set $T(x) := B(\infty) - B(x)$.

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$$\frac{T(\|\mathcal{Z}_t\|_{1+d})}{t} \rightarrow -\frac{s_0 \sigma^2}{2c_0} \quad \& \quad \frac{\mathcal{L}_t}{\|\mathcal{Z}_t\|_{1+d}} \rightarrow s_0^{-1}.$$

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Assume $\exists \sigma^2 > 0$ and $\exists s_0 < 0 < c_0$ such that, as $\|z\|_{1+d} \uparrow \infty$,

$$\sup_{z \in \mathcal{D}} |\operatorname{tr} \Sigma(z) - \langle \Sigma(z) e_x, e_x \rangle - \sigma^2| \rightarrow 0,$$

$$\sup_{z \in \partial \mathcal{D}} |\langle \phi(z), e_x \rangle - s_0| \rightarrow 0, \quad \sup_{z \in \partial \mathcal{D}} |\langle \phi(z), -e_y \rangle - c_0| \rightarrow 0.$$

Then $\mathbb{P}_\infty$ -almost surely as $t \rightarrow 0$ we have

$$\frac{T(\|\mathcal{Z}_t\|_{1+d})}{t} \rightarrow -\frac{s_0 \sigma^2}{2c_0} \quad \& \quad \frac{\mathcal{L}_t}{\|\mathcal{Z}_t\|_{1+d}} \rightarrow s_0^{-1}.$$

If $b(r) \approx ar^\beta$ (as $r \rightarrow \infty$) and $\beta < -1$ ($\iff B(\infty) < \infty$), then

$$\frac{\|\mathcal{Z}_t\|_{1+d}}{t^{1/(1+\beta)}} \xrightarrow{t \downarrow 0} (-s_0 \sigma^2 (1 + \beta) / (2c_0))^{1/(1+\beta)} \quad \& \quad \frac{\mathcal{L}_t}{t^{1/(1+\beta)}} \xrightarrow{t \downarrow 0} \tilde{s}_0 < 0.$$

Closing remarks

- This work extends results from [Pin09], who studied the case with normal reflection, $\Sigma \equiv \text{Id}$ and non-shrinking domains ($\beta \geq 0$), observing transitions between transience and recurrence only.
- Super/submartingale methods for lower bounds for tails of return times developed for the reflected diffusions are applicable to general continuous semimartingales and yield lower bounds on tails of invariant distributions in other contexts [BM26]. Both [BMW25] and [BCM26] rely on these ideas.

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Many questions about RBM Z remain open!!

Presentation sources

- BM26** M. Brešar and A. Mijatović. "Subexponential Lower Bounds for f -ergodic Markov Processes", *PTRF* 194(1): 997-1054 (2026)
- BMW25** M. Brešar, A. Mijatović, and A.R. Wade. "Brownian motion with asymptotically normal reflection in unbounded domains: from stability to transience", *AoP* 53(1): 175-222 (2025)
- BCM26** M. Brešar, J.P. Chavez Ochoa and A. Mijatović. "Brownian motion with asymptotically oblique reflection in unbounded domains: uniform ergodicity and descend from infinity", in preparation (2026)
- MMW23** M.V. Menshikov, A. Mijatović, and A.R. Wade. "Reflecting Brownian motion in generalized parabolic domains: explosion and superdiffusivity". *AIHP* 59(4): 1813-1843 (2023)
- MSW26** A. Mijatović, I. Sauzadde and A.R. Wade. "Central limit theorem for superdiffusive reflected Brownian motion". *Ann. App. Prob* 36(2):1840-1892 (2026)

See also my research YouTube channel [Prob-AM](#):

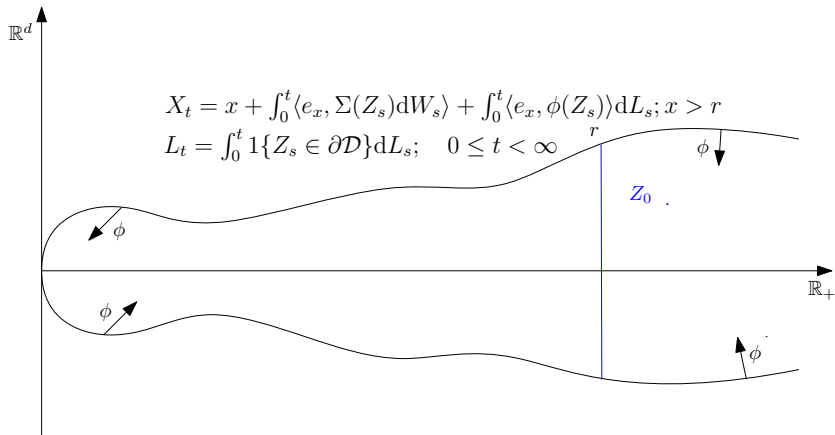


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- KR14** W. Kang and K. Ramanan. “Characterization of stationary distributions of reflected diffusions”. *Ann App Prob* (2014)
- LS84** P.L. Lions and A. Sznitman. “Stochastic differential equations with reflecting boundary conditions” *Communications on Pure and Applied Mathematics* Volume 37, Issue 4 (1984), pp. 511-537
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- SV71** D. Stroock and S.R.S. Varadhan. “Diffusion processes with boundary conditions”. *Communications on Pure and Applied Mathematics* Volume 24, Issue 2 (1971), pp. 147-225.

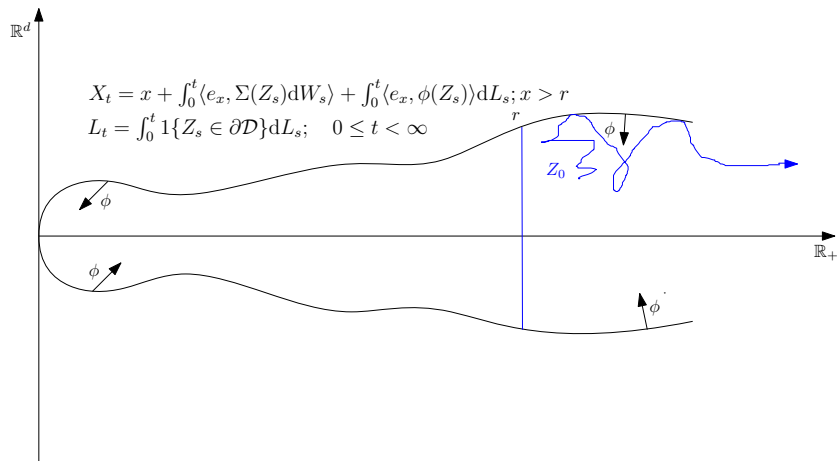
Structure of proofs

Our approach: deduce the long-term behaviour of Z from return times to the compact sets!



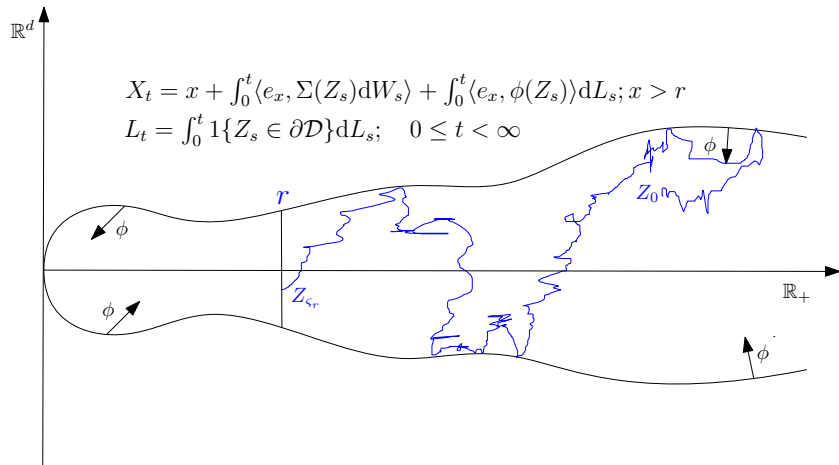
Structure of proofs - transience $\beta \in (\beta_c, 1)$

Recall $\beta_c = c_0 \sigma_1^2 / (s_0 \sigma_2^2)$ and $\varsigma_r := \inf\{t \in \mathbb{R}_+ : X_t \leq r\}$. We prove for every $Z_0 \in (r, \infty) \times \mathbb{R}^d$, the return time $\mathbb{P}_Z(\varsigma_r = \infty) > 0$



Structure of proofs - recurrence $\beta \in (-\infty, \beta_c)$

Recall $m_c := \frac{1}{2}(1 - \beta/\beta_c)$. We classify the tails of return time
 $c t^{m_c - \varepsilon} \leq \mathbb{P}_z(\varsigma_r > t) \leq C t^{m_c + \varepsilon}$.



Structure of proofs - positive recurrence $\beta \in (-\infty, -\beta_c)$

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What super/submartingale are suitable? **Inhomogeneity** of state space $\mathcal{D}$

A Lyapunov function

For any $\gamma \in \mathbb{R}$ and $w \in \mathbb{R} \setminus \{0\}$, let

$$f_{w,\gamma}(x,y) := (x + k_w)^\gamma \left(1 + w(1-w) \frac{\|y\|_d^2}{2(x + k_w)^2} \right)^{\gamma/w}.$$

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$$df_{w,\gamma}(Z_t) = \frac{1}{2} \Delta_\Sigma f_{w,\gamma}(Z_t) dt + \langle \nabla f_{w,\gamma}(Z_t), \phi(Z_t) \rangle dL_t + dM_t, \text{ for } 0 \leq t < \infty,$$

where

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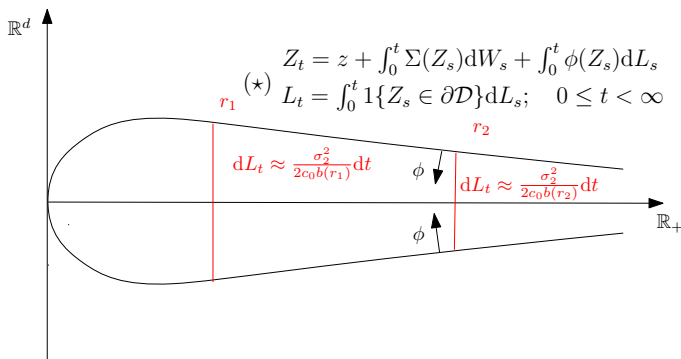
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- define $w_0 := 1 - s_0\beta/c_0$, then for large $x \in \mathbb{R}_+$ we have

$$\begin{cases} \langle \nabla f_{w,\gamma}, \phi(x, y) \rangle \leq 0, & \text{if } w > w_0, \\ \langle \nabla f_{w,\gamma}, \phi(x, y) \rangle \geq 0, & \text{if } w < w_0. \end{cases}$$

Technical difficulties

The super/submartingale criteria provide precise characterisation of **tails of return times**. But **ergodicity** requires also local behaviour, e.g. **Feller continuity** and **irreducibility**.



Feller continuity is a technical challenge. In a **shrinking** domain, **local time** accumulates **very fast**! Classical approach of [SV71] does not work.