INTERIM REPORT

on

PROJECT MATHS

J. J Grannell (Chairman) P. D. Barry M. Cronin
F. Holland D. Hurley

School of Mathematical Sciences,
University College Cork.

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Part I

Executive Summary

Executive Summary

The School of Mathematical Sciences at University College Cork, recognising its role in teaching Mathematics at third level and supporting teachers of Mathematics at all levels, established a Committee to examine *Project Maths*, the proposed new Mathematics syllabus at the Secondary School Level. We appreciate that Project Maths seeks to reform both *what* mathematics is taught in secondary schools and *how* it is taught. We also appreciate the labour pains of implementing such reforms. While there are aspects of Project Maths that we wish to endorse, we have points of concern with both the constituent character and practical implementation of Project Maths. We are concerned about all second level students, not just those who proceed to third level institutions and attend some mathematics courses there. With regard to the latter group, we are concerned about the extent with which they will be prepared for the depth and breath that is currently demanded upon commencing their third level Mathematical studies. In particular, we are deeply concerned about the following aspects of Project Maths.

- The exaggerated claims being made for the new approach including, a deeper understanding of mathematics and the acquisition of skills required for the development of a smart economy,
- The overwhelming emphasis being placed on a real-life context for study, and some of the examples which have already been developed for this purpose,
- The inadequate preparation of the teachers who are expected to teach the new syllabus,
- The unnecessary haste with which the new programme is being introduced into schools, coupled with the delays in providing syllabus details which is causing confusion and anxiety among teachers and students,
- The scarcity of suitable textbooks covering the proposed syllabus,
- The choice of material being dropped from the older syllabus.

Our recommendations can be summarised as follows.

Recommendations

1 We are very concerned about the influence of the PISA philosophy of mathematical education which seems to be heavily influencing Project

Maths. We believe that, in the introduction of a new approach to teaching and learning, a balance is essential and any innovation must be treated with caution and rigorously tested. In particular, we believe that the approach of the Singapore second-level Mathematics word problems should be used much more in contextual problems.

2 Mathematical Educationalists, participating in Project Maths, need to engage with Mathematical Scientists in developing a training programme which enables teachers acquire the necessary skills required in teaching the application of mathematics in various (albeit simple) contexts. There appears to be little understanding that training of this nature is required.

3 Serious in-service training should be provided as a major component of Project Maths. Weeks rather than days of training are needed. Third level mathematical scientists should be involved in the design and delivery of such in-service programmes. Furthermore, hands-on workshop activities should be central to these programmes.

4 The National Council for Curriculum and Assessment (NCCA) should seek the collaboration of the third level Institutions (Universities and Institutes of Technology) and the Irish Mathematics Teachers Association (IMTA) in the provision of summer and evening postgraduate courses in Mathematics appropriate to the Second-Level syllabus. Such courses should be designed to train a core of second-level teachers who would play a leading role in in-service programmes. Costs of tuition fees, travel and accommodation should be provided by the Department of Education and Skills. In addition, the third level mathematical scientists and mathematical educationalists should consider the provision of more extensive training to facilitate excellence in Mathematics teaching.

5 It will take time to evaluate the need for appropriate change in University matriculation requirements due to the evolving nature of the new Leaving Certificate courses and examinations. For matriculation purposes and to ensure that international standards are maintained, it is essential that the Universities require that specifically appointed

University examiners approve annually the draft examination papers and mark schemes.

6 There is an immediate need for a supply of textbooks to cover the material in Strands 1 and 2. We recommend that the NCCA, as a matter of urgency, encourage the production of such textbooks, both in print and in electronic form, and that they are complete and definitive. Improved teaching materials are also needed for all Strands, and at all levels, due to the novel non-standard nature of the new courses.

7 Examination questions must be formulated with mathematical rigour. They must be unambiguous and not open to misinterpretation. This is especially of concern in relation to context-based questions. The allocation of marks per part of each question in the examination papers should be made explicit.

8 Each Strand has been presented as a stand-alone component of the course. This is a serious mistake, and needs to be remedied at the first opportunity. Every effort should be made illustrate the interconnectedness of all the strands, and thereby give a coherent account of Mathematics. Proper guidance should also be given to the teachers as to the order in which topics should be taught. It makes no sense, for example, to teach (i) probability before the language of sets and counting techniques, and (ii) the normal distribution before the rudiments of differential and integral calculus.

Part II

Report

Introduction

Background

Mathematics is an intellectual discipline that has evolved over millennia - and continues to evolve - into a coherent body of knowledge that is admired as much for its aesthetic qualities as for its usefulness. Studied for its own sake, it provides the learner with an appreciation of the power of deductive reasoning and the facility to reach correct conclusions from firmly established facts or intuitive axioms. At its most basic level, it provides the user with a range of arithmetical skills to carry out daily commerce. At a higher level, it provides the language, principles and tools to enable the engineer and scientist to invent and construct all kinds of technology for scientific and societal advancement.

In recent years mathematics and mathematical education at second level have been the subject of both positive and negative comment in the Irish media. Positive aspects of this commentary focus on the pivotal role of mathematics in the scientific and technological developments which are required to support the Irish economy and its recovery. Negative aspects tend to focus on disappointing examination results and on the mathematics syllabus in second-level schooling. Particular concern has also been expressed about many teachers' weak foundation in mathematics.

Following from reports on post-primary mathematics education in Ireland and internationally, on the performance of Irish students in international mathematical tests, and the introduction of a new primary mathematics syllabus about a decade ago, the state Department of Education and Skills and the National Council for Curriculum and Assessment (NCCA) decided to undertake a comprehensive reform of second-level mathematics education. This reform involved syllabus, teaching methods and examining. The new system was given the name **Project Maths**. After some years of syllabus preparation, it was decided that Project Maths would be divided into five Strands. In 2008 a pilot scheme was launched in 24 selected schools based on *Strand 1: Probability and Statistics* and *Strand 2: Geometry and Trigonometry.* By 2010 some students in the pilot scheme had followed the material in years 1 and 2 and will sit the Junior Certificate examination in 2011; others had studied the material in years 5 and 6 and sat the Leaving Certificate examination in June 2010. In the meantime considerable teaching material was published and sample examination papers for the Leaving Certificate were tested. There were also some in-service courses for teachers.

Syllabuses for *Strand 3: Number*, *Strand 4: Algebra*, and *Strand 5: Functions* have been published and preparation of further teaching material is in hand.

Strand 1 and Strand 2 have been introduced in all schools as from September 2010.

Context of Interim Report

Project Maths is a radical departure from the existing syllabus, teaching methods, and examination style. Recognising the strategic role of the School of Mathematical Sciences at University College Cork in Irish mathematical education, the Head of the School set up a School Committee to study and report on Project Maths. Items of immediate relevance to our University are,

(i) the implications of Project Maths for our Matriculation and the various entrance requirements to our Faculties,

(ii) the consequences for our courses, and the associated resource implications, as a result of some sections of the old syllabus being omitted and new material being introduced.

Furthermore, the approach of Project Maths and the thinking behind it, is a subject of international debate in mathematical education. We wish to contribute to this debate in Ireland as part of our national responsibility.

In preparing this report, we have studied

(a) reports [1]- [6] by education groups, including the NCCA, on the current Irish second-level mathematics education,

(b) reports on second-level mathematics teaching internationally [7] - [21], especially those from nations which recorded the best performance at the Programme for International Student Assessment (PISA) and the Trends in International Mathematics and Science Study (TIMSS) examinations in recent times. Finland and Singapore, respectively, have been the best performing countries at these examinations.

(c) mainly the material produced online up to 31^{th} January 2011, that we are aware of, in connection with all five strands of Project Maths. [See *Appendix 9*.]

Apart from the syllabuses of the five strands, we have detailed information concerning teaching materials and examination formats and standards in the case of only *Strand 1-Probability and Statistics* and *Strand 2-Geometry and Trigonometry*. At this stage, therefore, we can issue only an Interim Report and we reserve our position until we absorb further material which is to be produced for the remaining three strands.

It is expected that some of our comments will be overtaken by further documentation produced by the Project Maths team.

Throughout this report, our comments, for the most part, are based more on earlier versions of Syllabuses and Assessment than on the online document titled ***Project Maths - Syllabus and Assessment for the Initial 24 Schools****.*

Summary View on Project Maths

Summary View

Efforts are ongoing to reform curricula for school mathematics throughout the world and Ireland is no exception. However there is scant reference in Project Maths documentation to the bitter international debate on the introduction of the various kinds of *reform mathematics* (some of which, apparently, stem from the theory of *situated learning* and *constructivism.* See *Appendix 5*). (We coin the word *context-constructivist* for use in this Interim Report to refer to combinations of these two approaches.) In the USA this debate is colloquially termed MATH WARS. The reform mathematics advocates largely consist of persons who are mathematical educationalists and the anti-side are mainly mathematicians and parents. Teachers are split between the two sides. After 15 years of strident debate in the U.S.A., the moderates there have been calling, since 2006, for agreement on an approach which combines the strong points of both the *reform* and *traditional* approaches. We shall refer to this here as a *combined* approach.

Our summary view is that what has been proposed and is being implemented in Project Maths is generally an attempt at a combined approach. We welcome this aspect of Project Maths. However, we believe there is an overemphasis on a context-constructivist approach and we favour different emphases in the details. We wish to make the following points.

Summary Comments

1. We would caution against the claims, and apparent assumptions, of better understanding of mathematics being acquired through the reform approach. We believe that much of the criticism of the traditional approach can be remedied through an up-skilling of teachers and a major reform of the examining system. It should also be noted that the experiences in other countries in the lowering of standards in essential material, resulting from the reform mathematics approach, has led to the international opposition to the reform approach. We need to be vigilant here and profit from the experiences of others.

2. We would also caution against unrealistic expectations as regards equipping pupils with ready skills to apply mathematical knowledge to the learning of other school subjects and to a variety of situations in life.

There is potential but, on the basis of what is known of the Project Maths programme, success can't be guaranteed. A considered and evidence-based approach and sustained effort will be needed to achieve this objective. (See [34] from which we take the quotations in *Appendix 6*.)

3. The PISA examining policy, which seemingly substantially influences Project Maths, is one which is aimed at testing 15-16 year-old pupils as to their mathematical literacy. By *mathematical literacy* we mean the mathematical competency they need to have to enter the work place, at an age at which compulsory education in Mathematics generally ceases across the globe, and perform efficiently and effectively as responsible citizens thereafter. A great proportion of the reform approach material deals with arithmetic[1] (including arithmetic applied to material in other Strands, see Appendix 13) and a considerable amount of this material has been implemented in our primary mathematics curriculum and teaching. On the other hand, the benefits derived from the TIMSS testing policy, which covers the same age range as does PISA but deals with considerably more algebra and geometry, seem to have been somewhat neglected in the design of Project Maths. For these reasons we think that the emphasis on context-constructivist methods should be less in the Senior Cycle than in the Junior Cycle. (See *Appendix 6*.)

   (We give detail of a comparison between PISA and TIMSS material in *Appendix 7*.)

4. The rolling nature of the introduction of Project Maths makes it difficult for the Universities to judge the suitability of Project Maths for Matriculation purposes on the basis of a Pass in Leaving Certificate Mathematics and the specific entry requirements of the different Faculties. It is also difficult to foresee which changes will be necessary in their courses resulting from the discontinuance of material which is in the previous school syllabus, or the inclusion of some new material. We believe that the Universities should make clear at this stage that they will insist that their approval be required of any revisions of the syllabus. Furthermore, we believe they should require that specifically

[1] by this we mean that body of mathematical knowledge which enables people to carry out simple everyday measurements, commercial transactions and motions from place to place.

appointed University examiners approve annually the draft examination papers and mark schemes.

5. The greatest difficulty we foresee in the successful implementation of Project Maths is the enormous burden of up-skilling that will be placed on teachers. From the point of view of the massive effort teachers will have to make in retraining, of the need to prepare much teaching material and sample examination material, and of the need gradually to gain experience of the whole process, we regret that the request of the Irish Mathematics Teachers Association (IMTA) (in a letter dated April 9^{th} 2009 regarding their concerns about Project Maths) for the postponement of the general implementation of Strands 1 and 2, until later than September 2010, was not acceded to.

6. In 2009, the Irish Educational Publishers Association (IEPA) lobbied the Minister for Education and Skills to delay the implementation of Project Maths until 2013. They argued that the timescale for the introduction of Project Maths over several years would be "cumbersome, confusing and unnecessarily expensive for families". In declining the request, the Department said that Project Maths had to be introduced "urgently" as part of a plan to create a "smart economy ". As a result, there is a lack of appropriate textbooks. Instead, teachers are being advised to use online material to teach the syllabus. This is hardly ideal: school administrators should not be expected to have to download large amounts of material, make multiple copies of it and distribute it to students.

7. Regarding the Examination papers we make the following remarks.

   (a) We are impressed by the Trialling Report [A9:37][2] on the pilot scheme concerning the sample papers, the marking schemes and the performance of the pupils. We noted the detail of questions drafted to achieve the general aims of Project Maths. We welcome the fact that all questions on each paper are to be answered for full marks. However we feel very strongly that, for Leaving Certificate papers, Contexts and Applications questions should be allotted at most 36% of the available marks.

   (b) The utmost care must be taken in formulating the questions; these should not be open to ambiguity. For example, rather than

[2]References of form [A9:**] refer to the list of documents listed in *Appendix 9.*

saying *Find the equation of the line passing through the points* $(-2,3)$ *and* $(4,5)$, it's preferable to say something like *Find the equation of the line passing through the points* $(-2,3)$ *and* $(4,5)$ *in the form* $ax+by+c=0$, *where* a,b,c *are integers.*

(c) The marks for each part of each question should be displayed on the examination papers and all questions should be weighted more or less equally.

(d) As mentioned in [25], PISA and TIMSS both use multiple choice questions, with these constituting a greater proportion of the total for TIMSS. We believe that our national examinations should not adopt this pattern of these two international tests without first giving the matter due consideration. It is our view that, rather than having questions requiring an identification of one of four given possibilities, it would be better that questions are phrased so that an answer is required to show some understanding.

(e) As for open-ended questions, we believe that they are unfair in timed examinations and they would be nightmarish to mark.

•• We have listed detailed comments on the examinations in *Appendix 3.*

For an outline of the structure of the syllabus, we refer the reader to *Appendix 11.*

8. We make the following comments on the syllabus.

   (a) There is little or no effort made to present the topics in the various strands as being interconnected parts of a coherent account of School Mathematics. No guidance is provided to teachers as to a natural order of the teaching of topics. For instance, substantial parts of Strands 3 and 4 should clearly be taught at an early stage.

   (b) Of all the strands, the syllabus for Strand 2 is by far the most coherent. There is a unity of purpose about it and in it students are introduced to the structure and logic of a branch of mathematics in a satisfying way. We appreciate that there has been a long tradition at school level of treating synthetic geometry more thoroughly than other branches of mathematics, but the manner in which it was designed should have served more as a template for material elsewhere in Project Maths.

(c) The syllabus contains no specification of methods to be used by students to formulate and solve context-based problems. However, half the marks in the Leaving Certificate examinations are being awarded for context-based material. Mature fields of application of mathematics have well-defined methodologies for simplification, abstraction, formulation and interpretation of mathematical problems in each field. In Project Maths, the decision was taken that mathematics would be applied in very simple contexts on the assumption that students would be familiar with the concepts of the context. However, it does not appear to be recognised that students (and teachers) must be instructed in the methods of application of mathematics, even in the simplest of contexts and it should not be left to them to discover these for themselves. (See *Appendix 6*)

(d) We are seriously concerned about the number and extent of topics which have been removed from the core material of the current syllabus and the extent to which this will impact on the teaching of Applied Mathematics, in particular, at both second and third level. Of especial concern in this regard is the decision to excise all references to vectors in the Project Maths syllabus. In the other direction, the introduction of new material, such as that in Strand 1, may present new opportunities and challenges at third level, especially to designers and instructors of business oriented programmes. (See *Appendix 4* for a list of deletions from the Higher Level Leaving Certificate.)

•• We have listed detailed comments on the syllabus in the various strands in *Appendix 1.*

9. We make the following comment on the teaching and resource materials.

   There are several sources of teaching materials available for the various Strands, ranging from outlines for teachers, the notes from short courses delivered to teachers to interactive IT applications for students. Each of these has its own merits but none can be described as a complete or definitive resource for either the teacher or the student. Apart from the topic Synthetic Geometry, when all of this material is combined it does not seem to cover the definitions and establishment of properties one usually expects in a textbook.

•• We have listed detailed comments on the teaching and resource materials for two of the strands in *Appendix 2*.

10. We make the following comment on problem-solving.

    Almost the whole thrust of PISA is to concentrate on 'authentic real-world problem solving' from which very ambitious, untested benefits are predicted to follow. This is a very important matter which deserves thorough discussion. We deal with it in *Appendix 5*, *Appendix 6* and 1(b) of *Appendix 13*.

Part III

Appendices

Appendix 1
Comments on the Syllabus

*Throughout this report, our comments, for the most part, are based more on earlier versions of Syllabuses and Assessment than on the online document titled **Project Maths - Syllabus and Assessment for the Initial 24 Schools**.*

• As a general remark, the wording *students should be able to*, appears in the syllabuses of all of the Strands. (See *Appendix 11*) The early teaching material in [A9:24 - A9:25] seems to be taking the interpretation *students should **only** be able to.* We believe that this is very restrictive. However, there are more general questions asked in [A9:26], pp 25 - 27.

Strand 1

The material in Strand 1 consists of core material and one of the optional parts of the previous syllabus but there is also inclusion of a considerable amount of new material. Up to now, the option of the previous syllabus, containing some of this material, was largely ignored by students, whereas now it's one of the five core components of the Project Maths syllabus.

1. The material in Probability should be delivered in sequence under the following headings: (i) Elementary set theory;(ii) Counting Principles;(iii) Mathematical Probability;(iv) Applications. Furthermore, material that needs techniques of calculus should be cross referenced to material in Strand 5.
2. Concerning Statistics, since properties of continuous probability distributions provide an important practical application of integral calculus we believe that these topics should be cross referenced.
3. The references to *lines of best-fit* without an explanation as to what is meant, in the section about *scatter-plots*, should be removed. A proper treatment of the *line of regression* should be given. Higher Level students should be taught, possibly somewhere else in the course,

how to determine the Line of Regression of a finite set of points (x_i, y_i), $i = 1, 2, \ldots, n$. (We outline in Appendix 8 how this could be done when the points don't all lie on a vertical line.)

4. Higher Level students should be taught that the mean $\bar{a}$ of n real numbers $a_1, a_2, \ldots, a_n$, minimises the expression

$$\frac{1}{n}\sum_{k=1}^{n} |a_n - x|^2.$$

This is a simple exercise in completing the square of a quadratic, and well within their capabilities. They will, of course, recognise the minimum value of this expression.

5. Higher level students should be taught that the median of n real numbers $a_1, a_2, \ldots, a_n$ minimises the expression

$$\frac{1}{n}\sum_{k=1}^{n} |a_n - x|,$$

but, as well, that it is not the only value of x that achieves the minimum value. This is only a slightly harder exercise, accomplished by first ordering the data in increasing order, say, and considering separately the cases when n is even, and when it is odd.

Strand 2

The bulk of Strand 2 consists of Synthetic Geometry, together with Coordinate Geometry, Trigonometry and Transformation Geometry. Further Geometry, which was an option in the old syllabus, has been deleted. The portion of Synthetic Geometry included in Project Maths for the senior cycle was absent from the Leaving Certificate programme for the past forty years; its return is to be welcomed.

The order of the subsections in Strand 2 is different as between Junior Certificate and Leaving Certificate. We use the latter ordering throughout.

§2.1 **Synthetic geometry**

If it is considered desirable that all our students be prepared as fully as possible for PISA testing, perhaps it should be pointed out that this requires that a small amount of basic material on solid geometry be included in the *Synthetic Geometry* section and that it would be better to move the applications of Theorems 7 and 8 from Ordinary Level Leaving Certificate to Junior Certificate; see, e.g., the problem on distance between houses discussed in *Appendix 5*.

§2.2. **Coordinate geometry**

For Junior Certificate, under *Description of topic* there should be the following entries:-

• *Lay out rectangular Cartesian coordinates for the plane, including axes, half-planes and quadrants*

• *Derive and apply coordinate formula for distance between two points*

• *Derive and apply coordinate formula for mid-point of two points*

• *Derive and apply the concept of the slope of a non-vertical line*

• *Derive and apply an equation of a non-vertical line in the form* $y - y_1 = m(x - x_1)$. *Derive and apply an equation of a vertical line.*

• *Derive and apply an equation of a line-segment*

• *Derive and apply equations of lines in the forms* $y = mx + c$ *and* $ax + by + c = 0$ *with* $(a, b) \neq (0, 0)$.

• *Derivation and application of the slopes formula for perpendicular lines*

• *Derivation and application of equations of a pair of parallel lines and of a pair of perpendicular lines*

• *Finding the point of intersection of a pair of intersecting lines*

• There should be an entry *Reading and interpreting graphs.*

• There should be an entry: *The graphs drawn and studied should include material on the piecewise-linear graphs that are used in business for once-monthly input of data*, and there should be corresponding entries under *Learning outcomes.*

§2.3 **Trigonometry**

• For Junior Certificate under *Description of topic* there is an entry *Trigonometric ratios.* This should read as *Derivation and application of the sine, cosine and tangent of acute angles. Solving trigonometric problems for right-angled triangles*, with a reference to the use of these under *Learning outcomes.*

• There is a huge jump as between Junior Certificate level and Ordinary Level Leaving Certificate in going from trigonometric functions of acute angles to trigonometric functions defined on $\mathbb{R}$.

There should be an intermediate stage to deal further with trigonometry of triangles. The following would be an appropriate entry :- *Extension of the definitions and properties of the sine and cosine to right and obtuse angles. Derivation and application of (i) the sine formula for a triangle, (ii) the cosine formula for a triangle and (iii) the formula* $\frac{1}{2}ab\sin\hat{C}$ *for the area of a triangle.*

• The following should also be added to Ordinary Level Leaving Certificate. *Specification of the angle of inclination* θ *in the upper half-plane of any line which passes through the origin and derivation of the formula* $m = \tan\theta$ *for non-vertical lines*, with follow on for the two of these under *Learning outcomes.*

• The entries - *define* $\sin\theta$ *and* $\cos\theta$ *for all values of* θ and - *define* $\tan\theta$ should be replaced as follows and either left at Ordinary Level or moved to Higher Level, *Introduce the concept of radian measure of a sensed angle (i.e. an anticlockwise angle), and then the concept of the sensed length* θ *of a winding path on the unit circle centered at the origin, starting from the point with coordinates* $(1, 0)$*. Define* $\sin\theta$ *and* $\cos\theta$ *for all* $\theta \in \mathbb{R}$ *and* $\tan\theta$ *for all appropriate values of* θ.

• In the Appendix of Trigonometric Formulae, it is not stated whether or not proofs of Formulae 1 to 9 are for functions defined (where appropriate) on $\mathbb{R}$.

• In [A9:31], page 13, there is an error in the proof where in relation to two points with coordinates (x_1, y_1) and (x_2, y_2), a horizontal distance is given to be $x_2 - x_1$ whereas it should be $|x_2 - x_1|$ and similarly a vertical distance is given to be $y_2 - y_1$ whereas it should be $|y_2 - y_1|$.

§2.4 **Transformation geometry**

For Junior Certificate, under *Description of topic* the whole entry consists of *Translations, central symmetry and axial symmetry.* The total entry under *Learning outcomes* is

- *locate axes of symmetry in simple shapes*

- *recognise images of points and objects under translation, central symmetry and axial symmetry (intuitive approach)*

• In [A9:27, page 4] there is a heading **Synthetic Geometry including Transformations**. This is inconsistent with other entries and should be broken into separate subsections named **Synthetic Geometry** and **Transformation Geometry**, respectively.

• For central symmetries, at the very least there should be *Proof that a central symmetry maps a segment onto a segment, and the image segment* *has the same length as the original segment. Corollary: A central symmetry maps each triangle onto a congruent triangle.*

• For axial symmetry, at the very least there should be *Proof that an axial symmetry in a line l maps a segment $[A, B]$ onto a segment $[A', B']$, and the latter has the same length as the original, treating separately the cases*

(i) $AB \parallel l$,

(ii) $AB \nparallel l, \ \ A \in l, \ \ B \notin l$.

At Higher Level, the same conclusion should be deduced from (ii) *under the separate conditions*

(iii) $AB \nparallel l$, A and B *on the same side of l as each other*,

(iv) $AB \nparallel l$, A and B *on opposite sides of l from each other.*

Deduce that each triangle maps onto a congruent triangle.

• The phrase *intuitive treatment* should be deleted.

• A little coordinate geometry should be introduced as that would make the study of these transformations easier.

• In the teaching materials for Transformation Geometry, e.g. [A9:23] and [A9:25], the examples seem largely based on using squared paper, with any motion being either horizontal or vertical and, in the case of axial symmetries these being only in axes of coordinates. This further erodes the standard of the material.

• We wonder if the following questions could be allowed?

Given distinct points A, B and C on a diagram, construct the image D of C under the central symmetry in which A maps onto B.

Given distinct points A, B and C on a diagram, construct the image D of C under the axial symmetry in which A maps onto B.

(See also *Appendix 12*.)

Strand 3.

Junior Cycle The title of this Strand is not descriptive of the material covered under this heading. (Indeed, we ask which sub-branch of Mathematics is called *Number*?) Something like *Applicable Arithmetic* would be more informative.

The content is very diffuse and consists of a hodge-podge of assorted topics whose purpose and logical connections are not made clear. Students are introduced to various number systems. However, the order properties of the real numbers and the absolute-value function are not even mentioned.

Primes are mentioned, but there is nothing about their role as the building blocks of the natural numbers. The section dealing with Indices contains a very poor treatment of the laws of exponents; it contains a list of formulae, many of which are meaningless. Treating the laws without proper definitions only sows confusion among students of all abilities. At the level of Junior Cycle, it suffices to deal with these laws for integral exponents.

Sets are discussed in Subsection 3.1. The usefulness of the language and notation of *Set Theory* in several topics of all Strands should require a treatment of this theory at a much earlier stage in the Syllabus.

Senior Cycle This is meant to consolidate students' knowledge of the real and complex number systems. They encounter Induction and the laws of logs and indices. As a device for crunching numbers, logs are out-dated and should be played down; their utility has long been replaced by the hand calculator. In any event, their properties are inherited from the exponential function, which is far more important, and should be thoroughly treated.

We feel that Ordinary Level students should learn to deal with geometric progressions. Higher Level students should, at least, be told about the fundamental factorisation theorem for natural numbers, that the primes form an infinite subset of $\mathbb{N}$ (Euclid's Theorem), and be given a proof of the irrationality of numbers like $\sqrt{2}$.

It's surprising to us that there is no mention of the polar representation of a complex number, or of de Moivre's formula in this section. Also, the rules for handling limits of sequences are not stated.

Strand 4

Junior Cycle This lacks coherence. We believe that students and teachers will have great difficulty understanding the material outlining what is algebra and then knowing how to implement it in a classroom.

Senior Cycle It's difficult to understand the introductory page: what's "relationship-based algebra"?

We believe Higher Level students should be shown a proof that a polynomial of degree $n \geq 1$ has at most n roots, counting multiplicities, and told of the relationships between the coefficients and roots of a polynomial.

They should also be shown how to reduce a cubic to normal form and how to decide when it has real roots. We expect students to be shown how to use the formula $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ to solve cubic equations like $8x^3 - 6x + 1 = 0$.

Ordinary Level students should be expected to be able to handle inequalities of the type $ax + b \leq cx + d$.

Strand 5

Junior Cycle According to the description of this Strand, students are expected to learn "to engage with concepts of function, domain, co-domain and range, and use functional notation". Surely, they should be taught such material much earlier in the course. They are also expected to plot the function 2^x before any meaning is given to the expression 2^x, when x is a real number.

This Strand is very lightweight and will require revision of the Applied Mathematics syllabus in the Senior Cycle.

Senior Cycle As we indicated in our remarks on Strand 3, undue emphasis is placed on logs, which are rarely, if ever, used nowadays for performing numerical calculations when the calculator is at hand. The following topics are omitted; Intermediate Value theorem, Mean value Theorem, and in Calculus, tangents to curves or speed and acceleration of moving particles, use of Calculus to sketch graphs of functions, Newton's Method, techniques of integration by parts and substitution. (See also *Appendix 4*.)

This material will not challenge Higher Level students.

Appendix 2
Comments on the Teaching and Resource Materials

Strand 1

Mathematics Resources for Students The learning outcomes listed for both Junior and Senior level in this document do not correspond with those listed in the syllabus, and we perceive that many of them are likely to arouse unrealistic expectations in the students' minds.

Formal definitions of terms (e.g., event, probability, unbiased, random), notations (e.g. P(A)) and methods (e.g. tree diagrams, multiplicative rule of probability, combinations & permutations) are not given here or elsewhere in this document.

There is no description of the different methods of assigning probabilities to events. It is not clear how some activities are to be completed; for example, activity 2.4. While there are innovative and stimulating activities, some activities are confusing, inappropriate or not even feasible; for example, activities 3.1 and Q.3 in Probability 7 for Senior Cycle, [A9:2].

Teacher Handbook This handbook for Strand 1 refers to material disseminated at two courses

(a) NCS-MSTL Summer Course in Statistics and Probability (2009)

(b) Statistics and Probability PMDT evening course (2010); modules 1-5).

Essentially it only provides teachers with outlines and references to resource materials rather than any material that could be used in a classroom.

The material from (b) is the most coherent collection of materials available. It appears to cover all the syllabus learning outcomes, though to varying degrees of detail. This should prove a very valuable resource for teachers, especially those who have attended the course, but may be of doubtful use to those who didn't attend. However, this material lacks strength in depth. In particular, more formal and rigorous introductions, descriptions and definitions of the topics, more worked examples, more exercises and their solutions are needed.

Teaching and Learning Plans 1-5 ([A9:7] to [A9:11]) These learning plans are currently incomplete in that only some of the topics/learning outcomes are addressed.

Material Created by Teachers These materials are naturally variable in terms of methods of presentation and quality. These should be seen as supplementary material for teachers, especially for those who are motivated enough to seek out the alternative presentations, examples/exercises that these materials would provide. The fact that such materials exist at this early stage of the implementation of Project Maths is concerning as it highlights the lack of a coherent set of materials for a teacher and/or a student, namely an official textbook-style presentation of materials or indeed a supply of commercial textbooks .

Specific Remarks about the Leaving Cycle Resource Material We discuss these under two headings.

Probability It's difficult to know how teachers can usefully teach a course based on the resource material for this strand, the presentation of which leaves a lot to be desired. (*Appendix 11* and Syllabus, Strand 1) Many of the claims made under the heading "Learning Outcomes", described in the various subsections of the document are likely to arouse unrealistic expectations in the students' minds. Using everyday words like "events", "likelihood", and "probability", without defining what they mean in a mathematical framework, will serve only to confuse students. To avoid confusion, these and other words such as "random" should be carefully defined, and/or replaced by "mathematical events", "mathematical likelihood", and "mathematical probability" throughout the Syllabus for Strand 1.

This section on Probability should be rewritten, and the essential concepts treated in a more logical step-by-step order. It should be delivered in sequence under the following headings: (i) Elementary set theory; (ii) Counting Principles; (iii) Mathematical Probability; (iv) Applications. Once (i) and (ii) have been covered, terms like "sample space, event, probability of an event" can be introduced as mathematical constructs thereafter, so that, from the outset, these will be viewed as mathematical concepts, and not have their everyday meanings. The concept of a tree-diagram should be developed and exploited more fully. Ideally, material involving the graphs and integrals of continuous random variables and density functions such as

$$e^{-|x|},\ -\infty < x < \infty; \qquad x(1-x)^2,\ 0 \le x \le 1;$$

$$e^{-x^2},\ -\infty < x < \infty; \quad \begin{cases} xe^{-x}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases},$$

should be handled in Strand 5.

Statistics Much of the material under the heading of Statistics 1 is unlikely to challenge good students. The significance of the measures “mean, mode and median” are not fully illustrated in the questions therein.

Scatter plots are dealt with under the heading of Statistics 4, where an attempt is made to define “a line of best fit ” when this can mean any line whatsoever. This section needs to be rewritten.

It’s hard to understand why Higher Level students at least are not told, though not necessarily in Strand 1, how to calculate the equation of the Line of Regression, which is presumably what is meant by “the line of best fit ”. After all, this can be found by determining the minimum of a sum of squares, a task that can be handled by “completing the square” of two quadratic polynomials. Students should know how to do this from their Junior Cycle course. At the very least, the formulae for determining its slope and y-intercept, should be displayed in the booklet *Formulae and Tables*. Knowing the appropriate formulae, the lines of regression for the data given in Q.1, Q.2, Q.3 and Q.4 in the section headed Statistics 4, can then be determined exactly with the aid of a calculator.

Suggestions for Strand 1

1. A mistaken impression is given that the three graphs depicted in Activity 3.1 of [A9:2] are representative, and that only one of three possibilities about the disposition of the mode, mean and median holds in general. This should be corrected. (We note that part (b) of Q.2 on last year’s LC paper could (only ?) have been answered by students who remembered the middle graph. This is unsatisfactory.)

2. Questions Q.1 to Q.9 pertaining to Activity 3.2 of [A9:2] involve calculating the mean, mode, and standard deviation of a finite number of data. Exercises involving the calculation of these for continuous distributions should be included in Strand 5.

Strand 2

1. We regard the teaching material prepared to date in [A9:23 - A9:25] as dealing largely with the simplest of material and the use of computer software graphics packages. It also seems to be tightly confined to what is listed under *Learning outcomes* which seems rather ominous.

2. There seems to be remarkably little in the way of open-ended problem-solving. Real life applications seem to be confined to draughtmanship and design for synthetic geometry and, surveying for trigonometry. In the material on transformations, all that is investigated under the various transformations, apart for the special case of enlargements, is the determination of the image of figures and whether certain measures are preserved. The document **Focus on Problem Solving & Strand 3, Number** [A9:36] is reproduced in *Appendix 5*.

3. Items [A9:26 and A9:27] should prove very useful for teachers. Items [A9:28 and A9:33] are essentially like [A9:26 and A9:27]. The large number of files is daunting and some effort should be made to harmonise the style of the titles.

All Strands

1. We regard the preparation of worthwhile **practical applications** to be difficult and recommend that class or group projects, which are teacher-led and supervised, be provided as part of the teaching material.

2. While accepting that material on open-ended problem solving will be taught, we think that it may not be suitable for actual examination in some of the Strands. In Strand 2 it does not seem to have been examined at Higher Level. At Ordinary Level it was the final part of Question 8 (pp.72-75), worth 5 marks out of 40, and the final part of Question 9(b) (pp. 83-87) and was worth 5 marks out of 45. We think that difficulty with this type of material could very well be disconcerting and upsetting for a pupil under examination conditions, while the low marks for it do not rate it as of great importance but the answers accepted for it do not warrant any more. We devote Appendix 6 to problem-solving.

3. We note that, for example, in [38] there are projects which are not intended for timed examinations.

Appendix 3
Comments on 2010 Higher Level Leaving Certificate Sample and Higher Level Leaving Certificate Papers

What follows are composite comments based on a perusal of the Higher Level LC Sample and Actual LC Mathematics Examination Papers used in 2010.

1. Precise formulation of questions is important. After all, if the examiner is allowed to use imprecise wording, why not the candidate? So, for instance, in part (a) of Q1, and part (a) of Q4, of the 2010 Sample HL LC paper, candidates should have been asked to find 'an equation' not 'the equation'.

2. Sometimes to avoid confusion in questions in which everyday words like 'random', 'biased' 'event' occur, it's essential to give the mathematical context explicitly. For instance, this should have been done in Q1 of the actual 2010 HL LC paper: candidates should have been told what P is, and its connection to A and B. In other words, the context should have been specified.

3. As far as possible, candidates should be asked to justify/verify their solutions to a question before being awarded full marks. So, full credit should not have been awarded in part (ii) of Q7 of the 2010 Sample HL LC paper, for giving the correct answer, especially when candidates were asked to calculate the correlation coefficient. An examiner should also assess the calculations.

4. "Real-world" problems should be formulated so that they are not silly. For instance, how realistic is the question about the lighthouse – whose presence is surely meant as a hazard warning, something that may not be known to all students in this day and age – on the actual 2010 HL LC paper? To answer part (c), one needs to know more about the location of the lighthouse; ditto for part (d): after all, we are not given any information about the coastline, and its relation to the lighthouse. (Couldn't the ship founder by coming too close to the lighthouse?)

5. Questions which invite students to 'estimate' an answer should specify the degree of accuracy required. Doing so, will also assist the marker.

For instance, this should have been done in Q7 of the 2010 HL LC paper.

6. Questions that rely on memory work only should not appear on any Mathematics examination paper. This is a major flaw with Q2 (b) on the 2010 HL LC paper. Even if students knew the definitions of mean, median and mode of a continuous distribution, they had no way of determining them from the graphs shown. Instead, to obtain full marks, they presumably would have had to recall similar graphs that appear somewhere in the Resource Material for Strand 1. (How else were they expected to answer this part?) Incidentally, the graphs and accompanying statements made there are misleading and don't cover all possibilities.

7. For examining and marking purposes, students should not be asked to *list* reasons (as they were in part (viii) of Q7 of the 2010 Sample HL LC paper,) why such and such lead to unreliable outcomes. Instead, they should be asked to identify them from a number of plausible/implausible stated reasons formulated by the examiner.

8. Students should not be penalised for using information/knowledge that they may have acquired by reading outside the course material.

9. It's ambiguous to ask for answers in 'standard form'. If a particular form of the answer is required for full marks, it should be clearly specified. Otherwise, the marking scheme is unfair. If – as they were asked to do in part (a) of Q4 of the 2010 Sample HL LC paper – candidates were expected to write their answer as $(0, 6)$, and lost marks for only giving $x = 0, y = 6$, the question should have read: 'Find the coordinates of D, and express your answer as an ordered pair.'

10. To assist teachers and examiners, more than one solution to a question should be provided, when this is possible

11. In part (iv) of Q3 of the 2010 Sample HL LC paper, a candidate received only partial credit for calculating the correct area, but not by deriving it from earlier sections. This was too harsh. In future papers, the word 'Hence' should be accompanied by 'or otherwise'. Furthermore, if students are expected to use the results of a previous part of the question, the following wording should be used: 'Using part (x) or otherwise', do such and such.

12. Q5 of the 2010 Sample HL LC paper was very lightweight, and not challenging at any level. Part (i) is trivial. Part (ii) is memory work. No justification was sought for part (iii). Full credit should have been been given in part (iii) only if it was clear that the candidate knew the meaning of 'range' and 'period'.

    Correct sketches are difficult to draw and, for marking purposes, the issue becomes too subjective to be left open. Instead, students should have been shown several possible graphs and asked to identify which of them best fits the graph of $y = 3\sin(2x)$.

13. Caution should be exercised when following the PISA philosophy to design contextual problems. For instance, one can detect the influence of PISA in Q7 of the 2010 actual HL LC paper! In its defence, at least a specific challenge is indicated in the LC question, unlike Example 1 of [35]. But one must ask: How can one determine the equation of the line, without knowing the data points?

Appendix 4
Topics removed from the previous Leaving Certificate Higher Level Mathematics Syllabus

We list the main headings of the old Mathematics Higher Level syllabus for the Leaving Certificate.

The core consisted of five main sections:

- Algebra
- Geometry—mainly coordinate geometry of the straight line and circle
- Trigonometry
- Sequences and series
- Functions and calculus
- Discrete Mathematics and Statistics.

In addition, students were offered 4 options:

- Further calculus and series
- Further probability and statistics
- Groups
- Further Geometry

Comment. The vast majority of students selected the *Further Calculus and Series* option in their answering of the Leaving Certificate Examination.

It was envisaged that students would be prepared in the core material and in one option.

The following topics have been removed from the core material:

- From **Algebra**: Factorisation of polynomials of degree 2 or 3; use of the notation $|x|$; solution of $|x - a| < b$; conjugate root theorem; nth roots of unity; identities such as $cos3\theta = 4\cos^3\theta - 3\cos\theta$; algebra of matrices of small order; vectors.
- From **Geometry**: Equation of line passing through the intersection of two lines; angle between two lines; parametric equations; plane vectors; transformation geometry.

• From **Trigonometry**: Sine and cosine rules applied to the solution of triangles; radian measure; use of result

$$\lim_{x\to 0} \frac{\sin x}{x} = 1;$$

inverse functions $x \to \sin^{-1} x$, $x \to \tan^{-1} x$ and their graphs.

• From **Sequences and series**: Sums of infinite series of telescoping type; informal treatment of limits of sums, products and quotients of sequences; recurring decimals as infinite geometric series.

• From **Functions and calculus**: informal treatment of limits of sums, products and quotients of functions; derivatives from first principles of e.g., $\sin x, \sqrt{x}, 1/x$; first derivatives of $\tan^{-1} x, \sin^{-1} x$; first derivatives of implicit and parametric functions; reference to points of inflection; Newton-Raphson method.

• From **Discrete Mathematics and Statistics**: difference equations.

The following topics have been removed from the optional material:

• From **Further calculus and series**: Integration by parts; the ratio test confined to power series; Maclaurin series for

$$(1+x)^a,\ e^x,\ \log(1+x),\ \sin x,\ \cos x,\ \tan^{-1} x;$$

series expansion for π.

• From **Further probability and statistics**: Nothing removed.

• From **Groups**: Everything removed.

• From **Further Geometry**: Everything has been removed, such as: locus of harmonic conjugates with respect to a circle; focus-directrix definition of an ellipse; transformations of the plane of the form $x' = ax+by+k_1,\ y' = cx+dy+k_2$, with $ad-bc \neq 0$; use of matrices; invariance properties; deduction of results for ellipse similar to those for a circle; similarity transformations, including enlargements and isometries, and their invariance properties.

• The **Algebra of Vectors** has been removed from the syllabus. This is regrettable and will present difficulties to teachers of Applied Mathematics at all levels – it is but one of the costs to be incurred as a consequence of revamping Geometry in its entirety, and increasing the amount of Probability and Statistics.

Appendix 5 Theories of Mathematical Education

Theoretical Input

Any person who dwelt only on the details of the syllabus for Project Maths, without examining the surrounding commentary, the resource and teaching materials, and the sample and extant examination papers, would not appreciate its really radical nature, and the influence exerted on it by existing theories of Mathematics Education.

These have been developed since the 1950s, and a general overview of those labelled Behaviorism, Cognitive Perspectives, Situated Cognition, and especially Realistic Mathematics Education (RME) can be found in [2]. Articles on individual ones of these are among the list of References.

In addition, since about 1990, international tests of second-level Mathematics (and science) have been in operation with the best-known two being TIMSS and PISA.

PISA examines 15-16 year-old students' *mathematical literacy* – the mathematical competency they need to have to enter the work place at an age at which compulsory education in Mathematics generally ceases across the globe, and perform efficiently and effectively as responsible citizens thereafter. (There are good summary accounts of the PISA Mathematics framework in [7] and [8].)

Project Maths seems to be heavily influenced in its approach and content by PISA, but we are not convinced of the merits of the PISA-type material. While it would be wrong of us to oppose totally the introduction of a new approach to teaching and learning school Mathematics, we think that a balance is essential, that any innovation must be treated with caution and rigorously tested, and that our national education in Mathematics must not go out on a limb for an approach which is not receiving general acceptance internationally. We are not alone in this view and share the concerns expressed by the authors of [31], an article which examines the role of cognitive psychology in Mathematics education. This article sounds a warning note about two movements namely, *Situated learning* which refers to learning having the context of the real, social world, with an emphasis on solving problems in what are deemed to be practical applications, and *Constructivism* which allows students, guided by a teacher, to explore and discover mathematical concepts and properties on their own. According to the authors, *some of the central educational recommendations of these* *movements have questionable psychological foundations. ... A number of the claims that have been advanced as insights from cognitive psychology are at best highly controversial and at worst directly contradict known research findings.*

Project Maths Output

To convey the relevance of Mathematics education theories to the implementation of Project Maths we first reproduce the opening paragraph of the section 'Teaching and learning' in the Syllabus overview of the Syllabuses for Strands 1 and 2, updated May 2009 which reads

In each strand and at each syllabus level, emphasis should be placed on appropriate contexts and applications of Mathematics so that learners can appreciate its relevance to their current and future lives. The focus should be on learners understanding the concepts involved, building from the concrete to the abstract and from the informal to the formal. As outlined in the syllabus objectives and learning outcomes, the learners' experiences in the study of Mathematics should contribute to their development of problem-solving skills through the application of their mathematical knowledge and skills to appropriate contexts and situations.

Secondly it's imperative for readers of this Interim Report to study the document [A9;36] **Workshop 3-Focus on Problem Solving & Strand 3, Number**, which we have added at the end of this Appendix for readers' convenience.

Thirdly we quote an entry from the syllabuses for several of the Strands:

Problem Solving and Synthesis skills

Students will be able to:

- *apply their knowledge and skills to solve problems in familiar and unfamiliar contexts*
- *analyse information presented verbally and translate it into mathematical form*
- *devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions*
- *explore patterns and formulate conjectures*
- *explain findings*
- *justify conclusions and communicate Mathematics verbally and in written form.*

Problem solving, word-problems, authentic context-based problems

Although mathematical problems have traditionally been a part of the mathematics curriculum, it has been comparatively recently that problem solving has come to be regarded as an important and prominent medium of teaching and learning Mathematics, with an initial phase emphasising teaching problem solving shifting to teaching via problem solving.

In this, and from the most elementary level, the term *word problems* became established and very widely used, although it is in fact an undefined term. A definition of this might be 'a word problem is a mathematics problem in which the main, or in any case the first, task of the solver is to extract an understanding/translation from a prose specification of it, into a mathematical language and symbolism which aids a solution of the problem'.

This often leads to equations and/or inequalities involving the variables inherent. Apparently, the most common types of word problems are ones about distances, age, work, percentages, mixtures and numbers.

A second large stream of problem solving emphasised *authentic context-based problems* in which the context intended was that of 'real world' or 'real life' situations. The word 'authentic' was used in PISA and elsewhere for problem solving which can create a context which simulates 'real life' and to be distinguished from less worthy 'word problems'. Arithmetic was always full of such problems but now the range is to be hugely enlarged.

Naturally there are divergences in terminology internationally, but that is the general picture, problem solving activity in two large streams; see e.g., [40]. However, Project Maths has shifted the overall phrase 'problem solving' to mean just the second stream of authentic context-based problems, and contrasts this with 'procedural ' solving which is regarded as a lower order of thinking skill to be encountered either in word problems or traditional treatments of mathematics.

This seems to be the basis of the language used in the 13-page document titled "Focus on Problem Solving, Project Maths Workshop 3", the purpose of which is unclear, which is interleaved in this report at the end of this Appendix. From its title one might expect to be given instructions, as discussed in [21] and [39] for instance, on how to solve problems, be they mathematical or non-mathematical. But, alas, this is not what the document is about. Readers will not learn one sensible idea about what strategies one should adopt to solve any kind of problem, nor be shown how to solve any illustrative ones. Mathematics teachers, in particular, the intended readership of the document, will not receive advice to how communicate problem-solving skills to their students. Instead, they will be asked to categorise certain mathematical problems, extracted from past examination papers, and listed in the first four pages of the document, into two kinds: ones that are 'procedural' and ones that are 'problem solving'. In the next three pages, they are expected to consider a series of questions about their own teaching strategies, their students' learning strategies, and rate them in a similar manner.

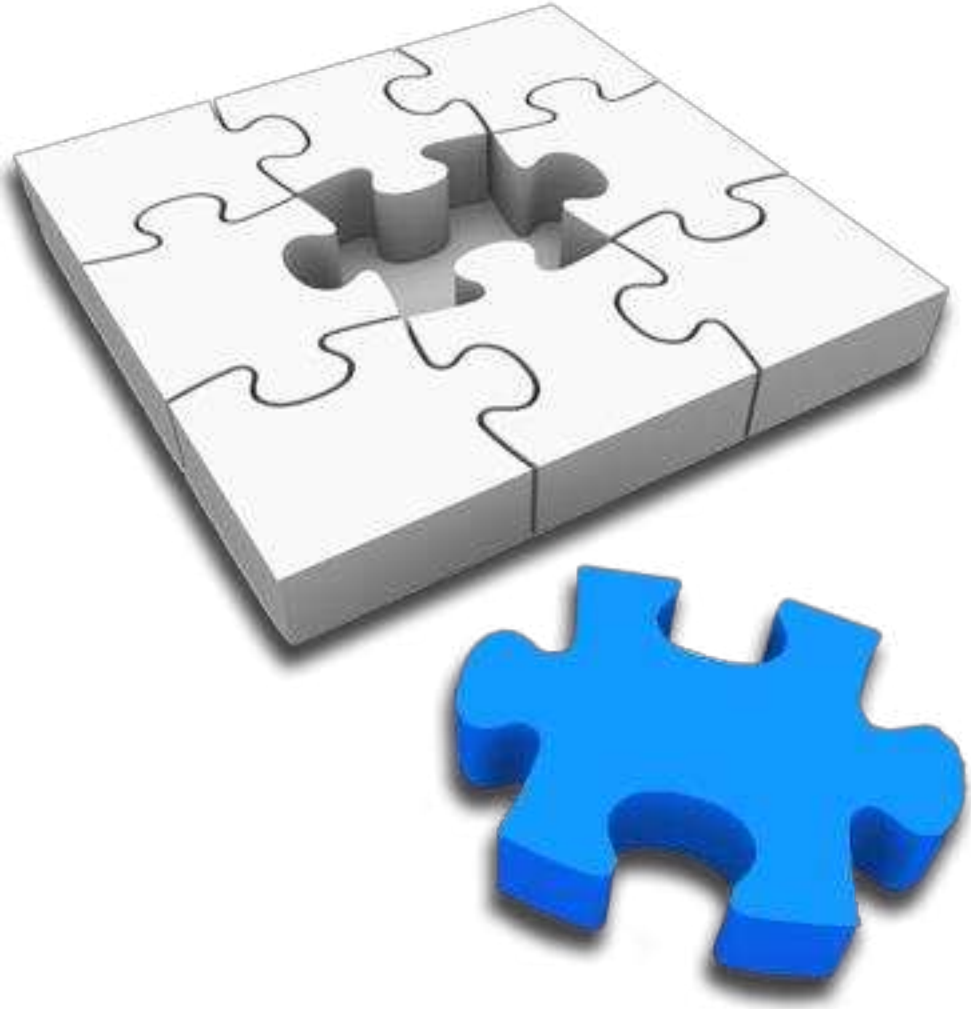
But in order to do these three things, they must first decide for themselves what is meant by a 'procedural' problem and one that is 'problem solving'; because these concepts are undefined in the document. In addition, readers of the document may be left with the impression that every problem is either 'procedural' or 'problem solving', which is surely incorrect, because, of course, questions can be of other kinds as well. While it may be possible to interpret what is meant by a 'procedural' problem—perhaps as one in which the solver can readily identify or be told explicitly what algorithm to apply to reach the desired goal—the idea of a 'problem solving' problem is one that lies outside the realm of Mathematics. The taxonomy of the family of questions belongs to another discipline.

The remaining pages lack coherence: on pages 10 and 13, respectively, information is given about five lines in the plane and scores on a test, respectively, with no stated objective; on pages 11 and 12, on the other hand, scatter plots of data are presented and questions—one of which relates to 'the line of best fit'—are posed for an unstated readership.

To sum up: the whole of this document serves no useful purpose for teachers and students of Mathematics, and deserves to be withdrawn.

Focus on Problem Solving

Project Maths Workshop 3



Name: ______________________

School: ______________________

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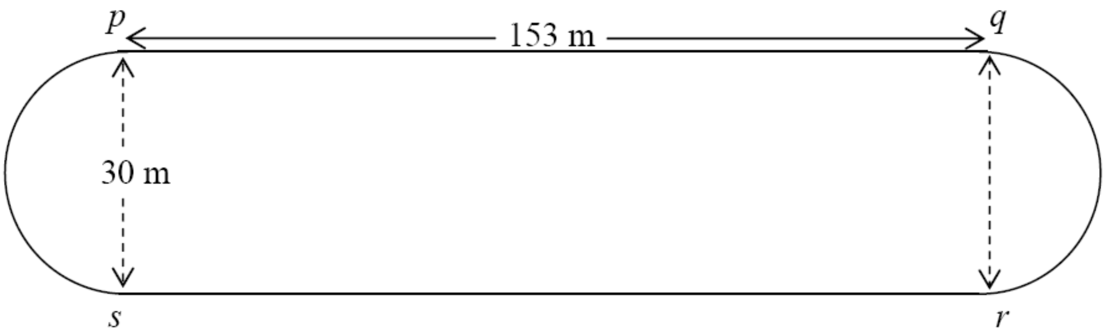
Development Team

WS3.1 – *Problem Solving*

Read through the past papers provided and tick in the boxes below whether, in your opinion, certain questions are procedural or problem solving.

JCOL 2008 Q1 (c) (i)	Procedural	
	Problem Solving	

An athletics track has two equal parallel sides [pq] and [sr] and two equal semi-circular ends with diameters [ps] and [qr]. $|pq| = |sr| = 153$ metres, and $|ps| = |qr| = 30$ metres.



Taking π as 3.14, calculate the length of one of the semi-circular ends, correct to the nearest metre.

JCOL 2008 Q1 (c) (ii)	Procedural	
	Problem Solving	

Calculate the total length of one lap of the track, correct to the nearest metre.

JCOL 2008 Q1 (c) (iii)	Procedural	
	Problem Solving	

Noirín ran a 5000 metre race on the above track in 15 minutes. Calculate, in seconds, the average time it took Noirín to complete one lap of the track during that race.

LCOL 1997 Q2 (c) (i)	Procedural	
	Problem Solving	

The length and breadth of a rectangle are in the ratio 9:5, respectively.
The length of the rectangle is 22.5 cm. Find its breadth.

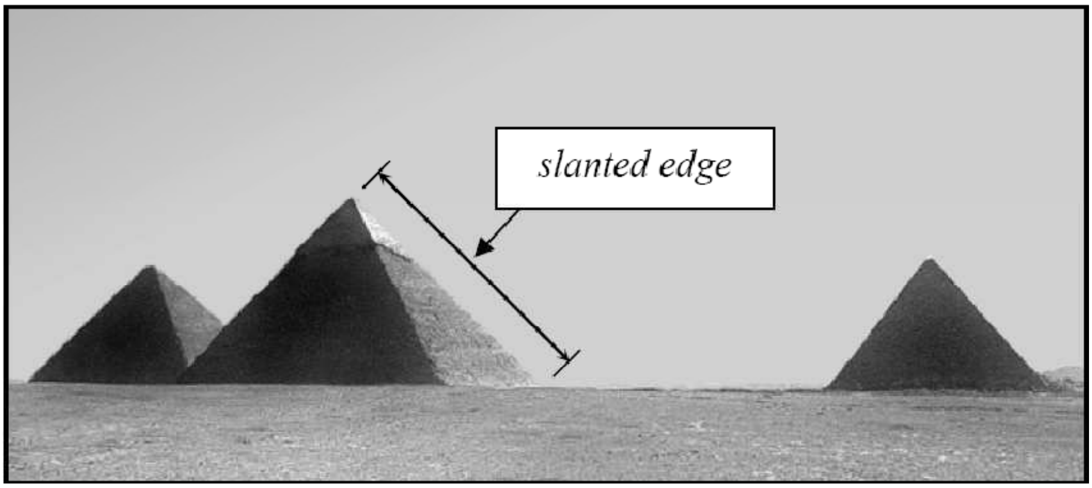
LCOL 1997 Q2 (c) (ii)	Procedural	
	Problem Solving	

Tea served in a canteen is made from a mixture of two different types of tea, type A and type B. Type A costs £4.05 per kg. Type B costs £4.30 per kg. The mixture costs £4.20 per kg.

If the mixture contains 7kg of type A, how many kilograms of type B does it contain?

LCHL 2006 Q5 (c) (i)	Procedural	
	Problem Solving	

The great pyramid at Giza in Egypt has a square base and four triangular faces. The base of the pyramid is of side 230 metres and the pyramid is 146 metres high. The top of the pyramid is directly above the centre of the base.



Calculate the length of one of the slanted edges, correct to the nearest metre.

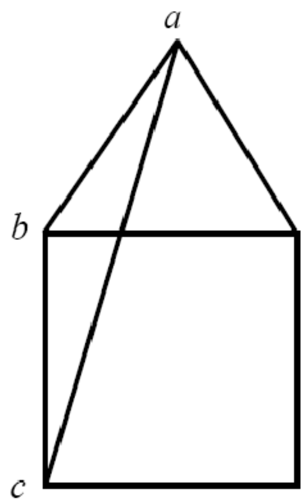
LCHL 2006 Q5 (c) (ii)	Procedural	
	Problem Solving	

Calculate, correct to two significant figures, the total area of the four triangular faces of the pyramid (assuming they are smooth flat surfaces).

JCHL 2005 Q6 (c) (i)	Procedural	
	Problem Solving	

The diagram shows an equilateral triangle and a square, each of side 6.
a is joined to c.

Find $|\angle abc|$ and $|\angle bac|$.



JCHL 2005 Q6 (c) (ii)	Procedural	
	Problem Solving	

Find $|ac|$, correct to one decimal place.

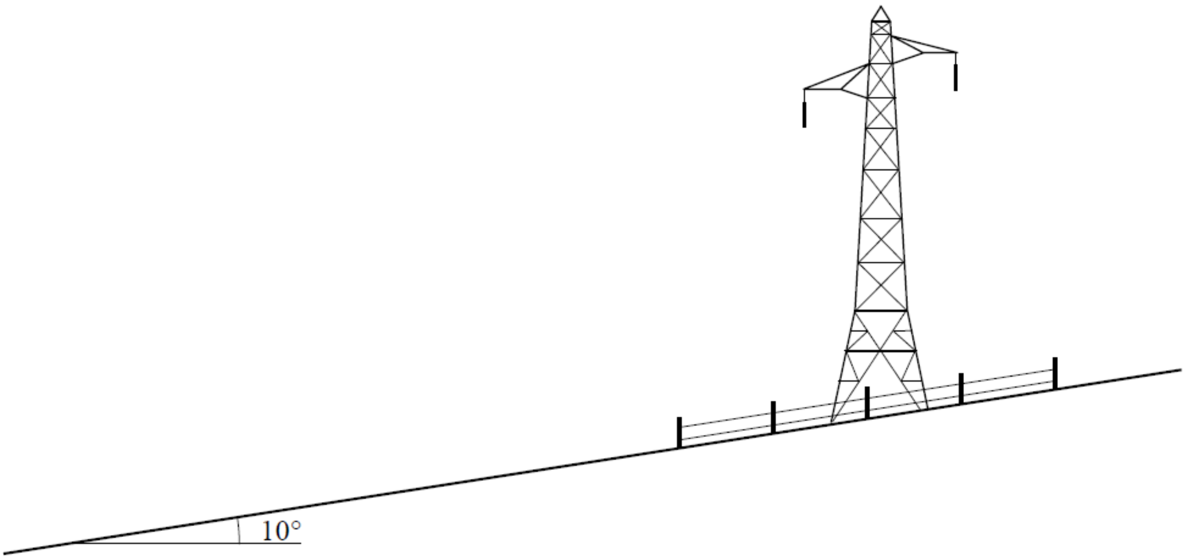
PMLCHL SAMPLE 2010 Q8 (a)	Procedural	
	Problem Solving	

Two surveyors want to find the height of an electricity pylon. There is a fence around the pylon that they cannot cross for safety reasons. The ground is inclined at an angle. They have a clinometer (for measuring angles of elevation) and a 100 metre tape measure. They have already used the clinometer to determine that the ground is inclined at 10° to the horizontal.

Explain how they could find the height of the pylon.
Your answer should be illustrated on the diagram below.

Show the points where you think they should take measurements, write down clearly what measurements they should take, and outline briefly how these can be used to find the height of the pylon.

Diagram:



Measurements to be taken:

Procedure used to find the height:

PMLCHL SAMPLE 2010 Q8 (b)	Procedural	
	Problem Solving	

Write down possible values for the measurements taken, and use them to show how to find the height of the pylon. (That is, find the height of the pylon using your measurements, and showing your work.)

JCHL 2008 Q4 (c) (i)	Procedural	
	Problem Solving	
In a certain week, *x* people shared equally in a club lotto prize of €2000. Write down an expression in *x* for the amount that each person received.		
JCHL 2008 Q4 (c) (ii)	Procedural	
	Problem Solving	
The following week, *x* + 1 people shared equally in the prize of €2000. Write down an expression in *x* for the amount that each person received that week.		
JCHL 2008 Q4 (c) (iii)	Procedural	
	Problem Solving	
In the second week, each winner received €100 less. Write down an equation in *x* to represent the above information.		
JCHL 2008 Q4 (c) (iv)	Procedural	
	Problem Solving	
Solve this equation to find the value of *x*.		

WS3.2 – *List 3 characteristics of questions that are problem solving questions*

1. ____

2. ____

3. ____

WS3.3 – *Reflecting on my Practice*

WS3.3A - *Teaching Strategies*

Think about your own teaching. Which of the statements below are true? Do they fit into 'procedural' or 'problem solving' approaches? Is there a balance between procedural and problem solving approaches in your classroom?

During lessons:	**Which of these statements are true?**	**Procedural**	**Problem Solving**
I begin with easy questions and work up to harder questions			
I ask questions with only one possible answer			
I always teach the whole class together			
I know exactly what will be done in the lesson before beginning			
I allow students to learn through doing exercises			
I give students manipulative models for hands-on discovery			
I get students working in groups discussing a new topic. I listen to them			
I pose questions with more than one answer			
I ask thought provoking questions			
I explain everything very carefully to avoid students making mistakes			
I allow students to consult a classmate, from time to time, when they are working alone			
I teach each topic from the beginning, assuming the students know nothing			
I teach each topic separately			
I ask students to think about how what they already know could help			
I ensure students use only the methods that I suggest			
I draw links between topics and move back and forth between them			
I facilitate students in discussing their mistakes			
I encourage students to explain to the class how they got an answer			
I always follow the textbook or worksheets closely			
I give the formulae or algorithms at the beginning of the lesson			
I teach one method only for doing each question			

During lessons:	Which of these statements are true?	Procedural	Problem Solving
I arrange pairs/groups of students to facilitate collaborative learning			
I get students to reflect on what they have learned from the lesson/lessons			
I get students to produce questions to examine the topic			
I find out what prior knowledge students already have and I don't teach those parts of the syllabus			
I allow students to compare different methods for doing questions			
I allow students to discover formulae and algorithms			
I welcome being surprised by the ideas that come up in a lesson			
I encourage students to work more slowly			
I give students the freedom to decide which questions to tackle			
I encourage students to invent their own methods			

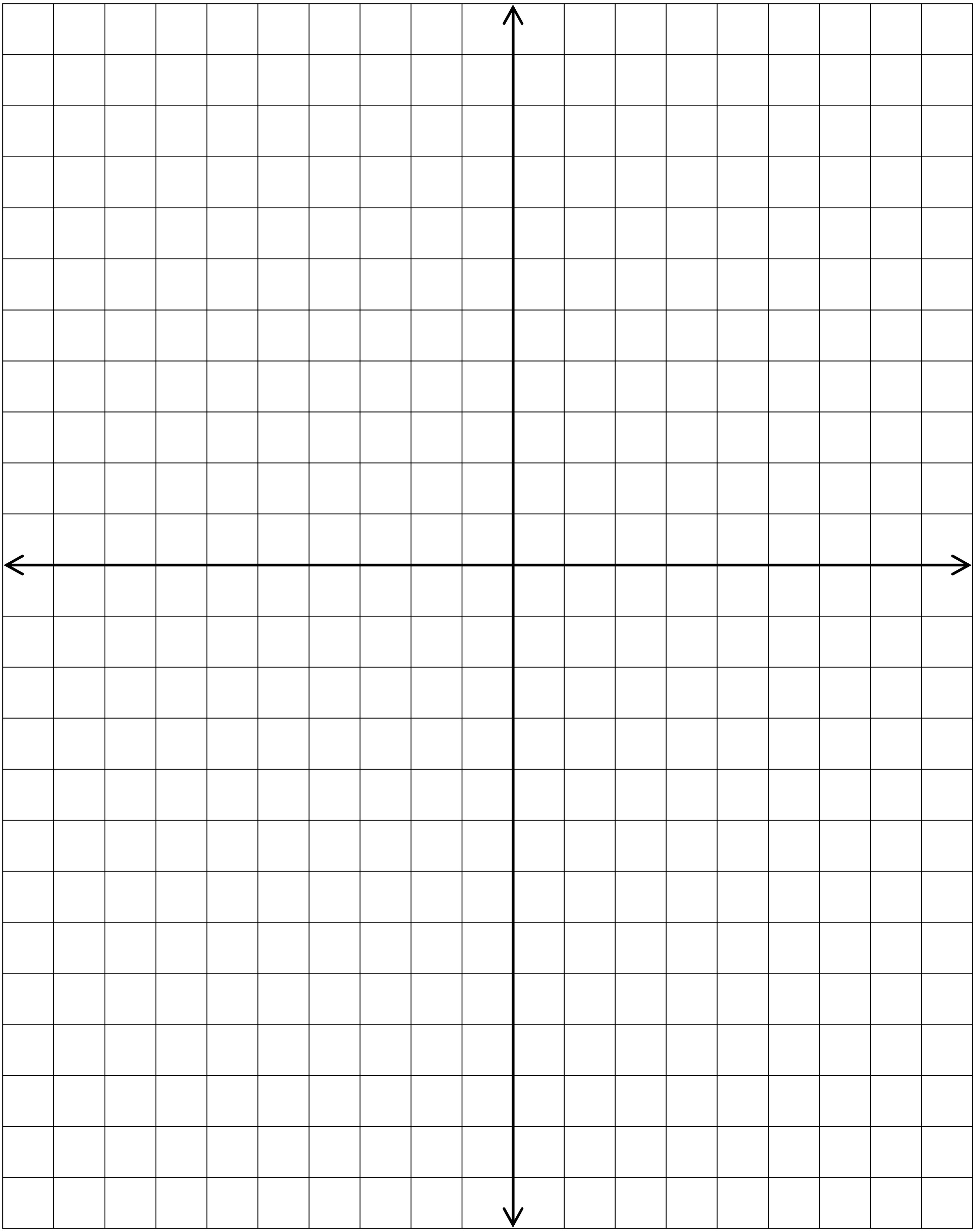
WS3.3B - Learning Strategies

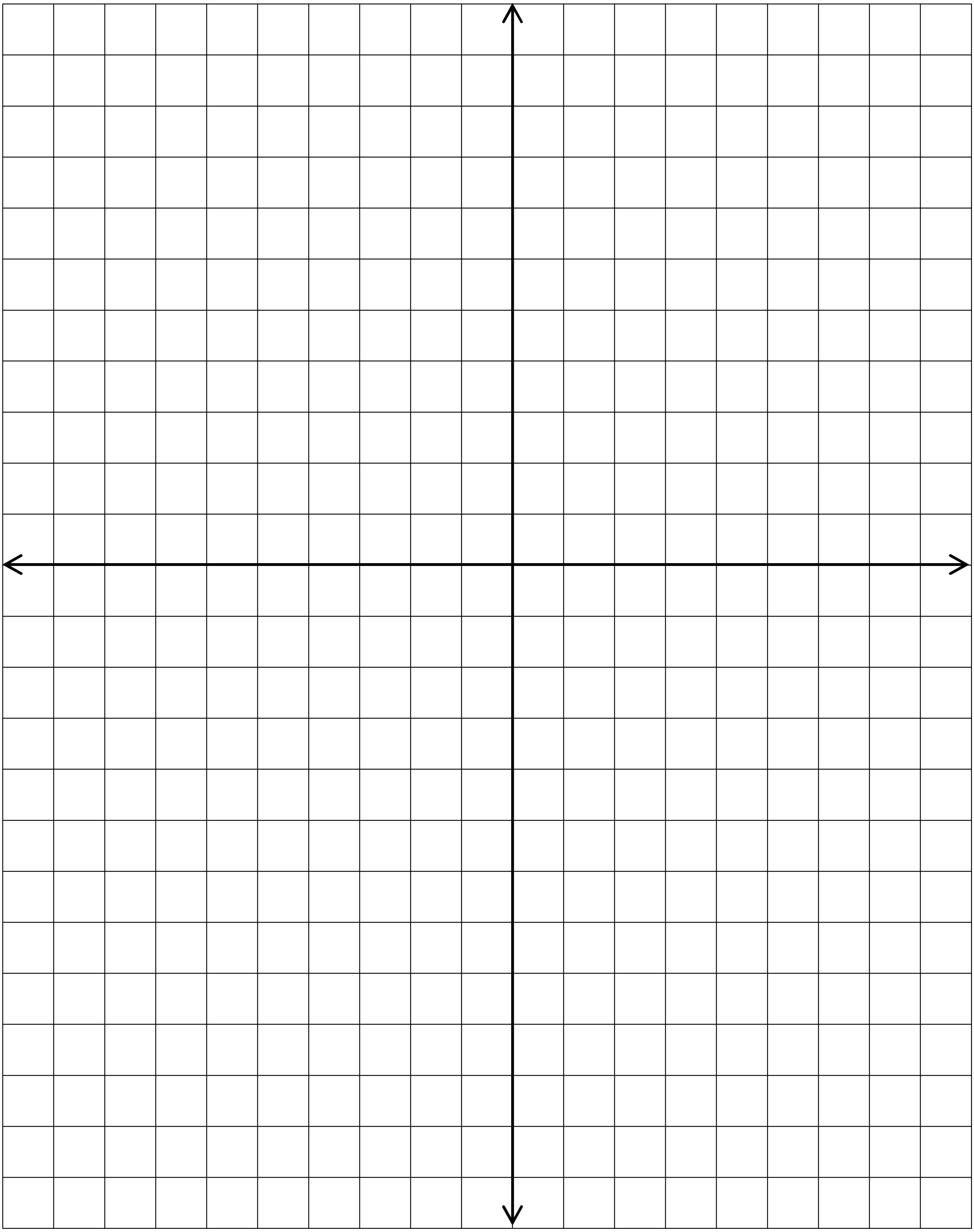
Now, think about what your students would say. Decide which of the following statements are true and into which column they fit.

During lessons:	Which of these statements are true?	Procedural	Problem Solving
I listen while the teacher explains how to do the question			
I copy down the method from the board or textbook			
I only do questions given by the teacher			
I always work on my own			
I use things/props to help me answer questions			
I try to follow all the steps of a lesson			
I only use the method(s) shown by the teacher			
I do easy problems first to increase my confidence			
I write out the questions before doing them			
I practise the same method repeatedly on many questions			

During lessons:	Which of these statements are true?	Procedural	Problem Solving
I wait until the teacher shows the method for doing particular questions			
I ask the teacher questions			
I try to solve difficult problems in order to test my ability			
I use diagrams/pictures when trying to solve a problem			
I get the opportunity to share and compare answers with other students in the class			
I don't give up when work is hard			
I discuss ideas in a group or with a partner			
I try to connect new ideas with things I already know			
I stay silent when the teacher asks a question			
I memorise rules and properties			
I look for different ways of doing a question			
I explain something to a classmate			
I explain while the teacher and my classmates listen			
I choose which questions to do or which ideas to discuss			
I make up my own questions and methods			

WS3.4 – *Graph Paper*



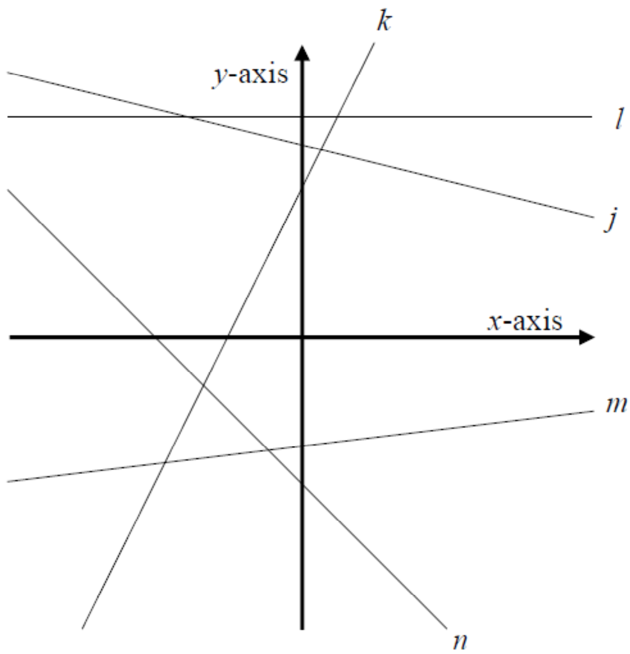


WS3.5 - *Assessment*

Question 6 (a) Ordinary Level

Five lines j, k, l, m, and n in the co-ordinate plane are shown in the diagram.
The slopes of the five lines are in the table below.

Slope	
2	
$\frac{1}{8}$	
0	
$-\frac{1}{4}$	
-1	



Notes:

Question 7 (b) Ordinary Level and Question 7 Higher Level

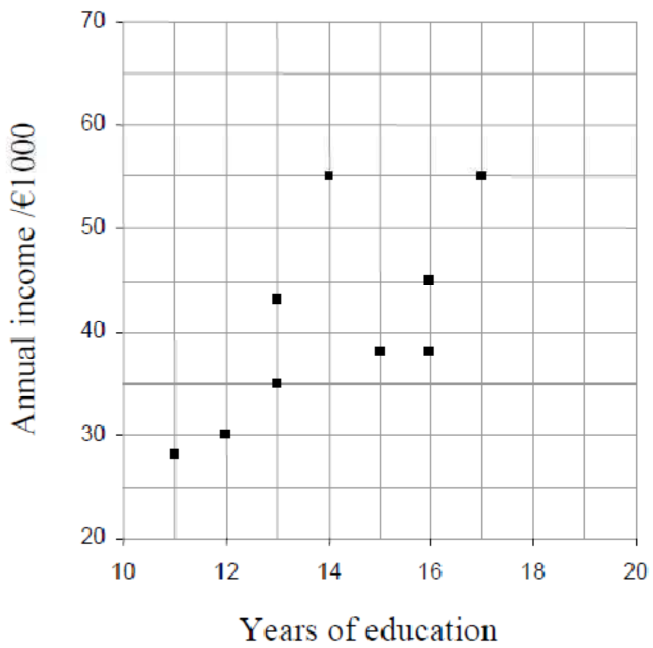
Ordinary level:

An economics student wants to find out whether the length of time people spend in education affects how much they earn. The student carries out a small study. She asks twelve adults to state their annual income and the number of years they spent in full-time education. The data are given in the table below, and a partially completed scatter plot is given.

Higher level:

An economics student is interested in finding out whether the length of time people spend in education affects the income they earn. The student carries out a small study. Twelve adults are asked to state their annual income and the number of years they spent in full-time education. The data are given in the table below, and a partially completed scatter plot is given.

Years of education	Income /€1,000
11	28
12	30
13	35
13	43
14	55
15	38
16	45
16	38
17	55
17	60
17	30
19	58



Ordinary Level

(i) The last three rows of data have not been included on the scatter plot. Insert them now.

(ii) What can you conclude from the scatter plot?

(iii) The student collected the data using a telephone survey. Numbers were randomly chosen from the Dublin area telephone directory. The calls were made in the evenings, between 7 and 9 pm. If there was no answer, or if the person who answered did not agree to participate, then another number was chosen at random.

Give **one** possible problem that might make the results of the investigation unreliable.

State clearly why the issue you mention could cause a problem.

Higher Level

(i) The last three rows of data have not been included on the scatter plot. Insert them now.

(ii) Calculate the correlation coefficient.

(iii) What can you conclude from the scatter plot and the correlation coefficient?

(iv) Add the line of best fit to the completed plot above.

(v) Use the line of best fit to estimate the annual income of somebody who has spent 14 years in education.

(vi) By taking suitable readings from your diagram, or otherwise, calculate the slope of the line of best fit.

(vii) Explain how to interpret this slope in this context?

(viii) Same as first paragraph in (iii) ordinary level.
List **three** possible problems regarding the sample and how it was collected that might make the results of the investigation unreliable. In each case, state clearly why the issue you mention could cause a problem.

Notes

Question 3 (b) Foundation Level

Seán's French teacher gives tests that are marked out of 10.
Seán got the following results in five tests:

7, 5, 6, 10, 7

(i) ____

____

(ii) Áine got the following results in the same five tests. She was not in for the fourth test.

8, 5, 7, – , 7

____

____

Notes

Appendix 6 Comments on PISA Test Examples

Comments on context-based problem solving in familiar and unfamiliar contexts.

The application of Mathematics, even in the simplest of real-world contexts, involves making idealizations. In classical areas of application of Mathematics, methods of simplification have been developed which result in simple mathematical problems which we can solve. Some of these methods of simplification have taken centuries to develop - mechanics is a good example of a source of such challenges. Those who study the application of Mathematics in this field learn methodologies of idealization that have been found to be successful. After exposure to a range of problems in this context, students gain confidence in the application of the established methodology. In posing problems to such students, it is not necessary to spell out everything in detail when specifying the problem for them, as they are experienced in applying standard methods of idealization. Biology is a more recent application area of Mathematics. Remarkably, some principles of idealization from the world of mechanics can also be applied in Biology. However, Biology brings new challenges and a new methodology has developed in that field. Again, with experience, students can gain confidence in applying the new methods of idealization specific to Mathematical Biology. In addition to methods of idealization, students of mechanics or Biology, must learn something about the subject to which they are applying Mathematics.

A fundamental idea in Project Maths is to apply Mathematics to everyday problems which the examiners can assume are familiar to students. Thus, it is assumed that they do not need to learn about the application area. However, the syllabus in Project Maths, unlike the syllabus in the classical areas of application of Mathematics, is silent on methodology of idealization. This is a flaw. There is a clear assumption that teachers, and also students, can invent the methodology themselves. This is an error - such an expectation is unreasonable. Students are expected to formulate and solve problems in unfamiliar contexts. To do this they would need a methodology of idealization that could be applied to any problem they might encounter. Textbooks containing methods of idealization for the application of Mathematics in specific contexts, such as mechanics or Biology, abound, but there are no textbooks, at secondary level, containing a methodology for applying Mathematics to anything that is within the compass of students' experience.

Project Maths, in common with PISA and TIMSS, appears to envisage tackling, at once, all components of context-based problem solving. This approach could be quite challenging for both teachers and students. An alternative approach would be to break the application of Mathematics into at least two steps. As a first step, students could be introduced to word-problems which do not necessarily have anything to do with reality. The problem designer would think of a mathematical problem and invent a word problem which reduces to this mathematical problem. To solve such a word problem, students must identify the relevant mathematical unknowns, assign mathematical symbols to them, formulate the mathematical equations by translation of the given word equations, solve them, and convert the solution into words. Solving problems of this type is challenging. Students who have mastered this are in a very strong position to tackle the next step: solving real problems. The new skill to be learned consists of idealization of a real problem to get a manageable mathematical problem. There is no single solution to this challenge. Properly trained teachers could guide students through this process in class or supervise projects in this area. In a timed examination context however, it would be wise for the examiners to complete the idealization step to yield a word problem and ask the students to solve the word problem. If a problem familiar to all students were to be posed, and for which the idealization methodology is familiar to the students, then it would be reasonable to expect them to carry out the idealization step also.

There appears to be an expectation underlying Project Maths that teachers can invent or discover real world applications of Mathematics, and implicitly, can develop the idealization methodologies themselves. This is unrealistic. The development of methodologies to tackle real-world problems, however simple, in a range of contexts at second level is, in fact, a challenge to both mathematical scientists and mathematical educationalists. It is not even clear that the aim is achievable. Thus, at this stage, we would urge caution in assigning a large number of marks for context-based problems in the State Examinations. Exhortations to teachers to place more emphasis on context-based problems, along the lines of PISA, while well-intended, are misplaced [4].

Note also '*The Challenges: how realistic is real world maths*?'in [2, pp 115 - 119] and 1(b) of *Appendix 13*.

Examples

Three examples which appear in PISA Framework Reports raise serious concerns about the understanding of their authors and examiners in relation to applying Mathematics in real world contexts and about the manner in which students' work is assessed.

1. **Example on Streetlight** [34, p 26]

   Problem *The Town Council has decided to construct a streetlight in a small triangular park so that it illuminates the whole park. Where should it be placed?*

   Neither a student nor a mathematician could solve this problem.

   A mathematician's immediate response to this is that the Council has not provided sufficient information to gain a clear understanding of the problem. In a real-life situation, a dialogue would take place between the Council and a mathematician to determine unambiguously what is the problem.

   The authors of the report provide a detailed explanation of how they would 'solve' the problem claiming that their approach follows '*the general strategy used by mathematicians*' and provides an example of ' *how informed and reflective citizens should use Mathematics to fully and competently engage with the real world*'. In fact, the authors make errors in formulating and 'solving' the problem, failing to recognise whether or not adequate information has been provided to identify the Council's problem and they proceed to make assumptions about details supplied by the Council. For example, they disregard the Council's requirement that the light be placed in the park by allowing it to be placed outside it, and they disregard the requirement that the light be a streetlight and thus should be located on a street.

   A competent mathematician tasked with addressing the Council's problem would certainly not proceed in the manner specified by the authors, nor should any sensible citizen.

2. **Example on distance between houses** [35, p 26 and 36, p 111]

   Problem *Mary lives two kilometres from school, Martin five. How far do Martin and Mary live from one another?*

   Neither a student nor a mathematician could solve this problem.

The answer to the question cannot be computed as not enough information has been provided. A student who makes this observation should get full marks.

It is pointless to imagine what the real problem might be. The problem may be very easy - Mary and Martin's houses and the school could all be on one side of a straight flat road and the problem reduces mathematically to addition or subtraction of real numbers. However, the problem may be very difficult - the houses and school may be located in hilly countryside surrounded by fields, walking through fields may/may not be allowed, or the distance from Mary to Martin's house may be the geodesic distance joining them - then a problem in differential geometry, far beyond the competence of 15 year olds to solve. The authors, in their commentary, recognise that the problem is ambiguous and say that teachers they consulted offered four different interpretations of what was being asked. They favour the view taken by one group of the teachers:

'*A small group thought it was an excellent item because one must understand the question, it is real problem solving because there is no strategy known to the student, and it is beautiful Mathematics, although you have no clue how students will solve the problem.*'

The authors clearly expect the students to imagine what the problem might be, but seem not to realise the vast range of complexity in the environments that students might imagine. This concept of taking a problem statement and of imagining how to supply the missing information in order to formulate a mathematically well-defined problem is in direct contradiction to the scientific method. There is long-standing terminology in higher mathematical science of 'well-posed problems' and 'ill-posed problems'. 'Open-ended problems' are just the latter under another name.'

The authors of the report give no information on how they would mark student's answers to such questions. We believe that it is simply unacceptable to give test questions that are open to misinterpretation or for which answers of widely differing levels are possible.

3. **Probability example** [34, p 100; 35, p 87 and 36, p 97]

Problem *If two fair dice have been rolled and one of them shows four, what is the chance that the sum exceeds seven?*

The answer the authors provide, (50%), is incorrect. Assuming that all the results are equiprobable, as the dice are fair, the probability is 5/11 and not 1/2 as claimed by the authors of the PISA reports: 2003 [34], 2006 [35], and 2009 [36].

Whatever erroneous reasoning the authors use, it is alarming to see such an error appearing in three successive PISA Framework Reports.

Appendix 7 Comparison of PISA and TIMSS

We look at comparisons of PISA and TIMSS to shed light on the former.

1. For a general comparison we refer to Hutchison and Schagen [29], which deals with science as well as Mathematics. The respective frameworks of TIMSS and PISA are given on pp. 1-6, an account of their test items on pp. 10-13, and details on testing, as well as conclusions, questions and recommendations on pp. 25-29.

   Comment We are puzzled that Ireland has not been entered for a TIMSS test since 1995 and wonder why.

2. One aspect of the results of comparison which we wish to emphasize is that **there is a substantial amount of ordinary school Mathematics which is not included in PISA** [4, p 30].

   This can be inferred from the information on test items in [29] but is explicitly stated by Margaret Wu in [24]. She remarks that in the comparison of PISA 2003 Mathematics and TIMSS Grade 8 Mathematics assessments, 42 of the 99 items, for the latter, were deemed not fitting the PISA test. In particular, many geometry and algebra items in TIMSS do not appear in PISA tests.

   We note that some proponents of PISA material claim that Mathematical Literacy is exactly what should be taught to students up to the end of junior level secondary school. For a contrary view we refer to Grønmo and Olsen [30]. In their final paragraph, they state that as a result of their analysis and comparison between TIMSS and PISA, they believe that *in order to do well in daily life Mathematics students need a basis of knowledge and skills in pure Mathematics, especially elementary knowledge and skills in numbers. This indicates that it is important in school curriculum that mathematical literacy is not seen as an alternative to pure Mathematics. A reasonably high level of competence in pure Mathematics seems to be necessary for any type of applied Mathematics.* Continuing, they say that *if too little attention is given to the full cycle of applied Mathematics, it is unlikely that students will develop the type of competence we may call mathematical literacy.*

3. As Finland has the best record in PISA tests, we refer to Martio [7], Malaty [8] and [41] for views from Finland on Mathematics teaching there.

Appendix 8
The Singapore Mathematics Framework

In the last decade both Finland and Singapore have featured prominently in one or other or both of TIMSS and PISA, on which their students have had marked success in Mathematics. As a result, the reasons for their successes in this subject have been scrutinised by educators in other competing countries, anxious, no doubt, to emulate the achievements of Singapore and Finland. However, while this may be a laudable objective, it needs to be stressed that success in such tests is not necessarily indicative of students' understanding of a particular Mathematics curriculum, nor, indeed, should such a curriculum be designed to attempt to ensure success at them.

In Finland's case, attention has largely focused on the training and competency of the Finnish Mathematics teachers (all of whom have a master's degree in Mathematics), the freedom they appear to have to design and implement their own syllabus, and the respect they enjoy in their professional careers [7 - 9]. In the case of Singapore, success in the aforementioned international tests can be traced to a two-pronged approach, namely, the problem-solving strategies inculcated in the students at the primary level, and the pre- and in-service training of the teachers. Here, we wish to allude briefly to the problem-solving strategies employed by the Singaporean Mathematics teachers.

The Singapore approach

After its independence in 1965, Singapore embarked on an effort to educate its citizens to a high standard, and over time its Curriculum Development Institute developed a Primary Mathematics program, which has attracted the attention of educators worldwide. In particular, it has found favour with many American Mathematics teachers, who have adopted a similar approach and use Singaporean textbooks in the classroom. An American advocate of the system is Bill Jackson, a school teacher, who has described its essential features in his four-part blog "Singapore Math Demystified" [11 - 14], to which we refer the interested reader.

The approach to mathematical problem-solving as practised in Singapore is built around a philosophy that incorporates five key components, namely: " Concepts, Skills, Processes, Attitudes and Metacognition" which are adumbrated by Jackson less succinctly in Part 3 of his blog. These steps are deemed so central to The Singapore Mathematics Framework that they are the names displayed on the sides of its pentagonal logo. At the primary level, these are the guiding principles by which teachers proceed from the concrete to the abstract, with the aid of visual aids, called bar models, which have their origin in Euclidean geometry.

Whether by accident or design, these principles bear a strong resemblance to the sequence of five steps that were identified in the mid-1970s by Anne Newman, an Australian language educator, that she thought were necessary to solve a mathematical word-problem, and which she called: "Reading (or Decoding), Comprehension, Transformation (or Modelling), Process Skills, and Encoding" [19, 20].

In turn, these are very similar to the four steps that, as far back as the mid-1940s, George Pólya advised one should follow to solve mathematical problems, and promulgated in his book [21], a copy of which should be in the hands of any person teaching or interested in the art of problem-solving. With wit, clarity of exposition, and brilliantly chosen examples, mainly from the realm of Euclidean geometry, Pólya explains why it's essential to (i) *understand the problem*; (ii) *devise a plan for its solution*; (iii) *execute the plan* and (iv) *reflect on the outcome*. But he was well aware that these principles could also be applied to solve other problems, even puzzles, and gave cogent examples to illustrate his thesis. But more to the point, according to Andy Clark [17], authors of Singapore textbooks are very familiar with Pólya's 4-step model, and include it, in particular, in manuals designed for teachers.

Appendix 9 List of Project Maths Resources

We have not checked all the teaching and resource material listed here.

A9:1 **Mathematics Resources for Students, Junior Certificate - Strand 1, Statistics and Probability**.

A9:2 **Mathematics Resources for Students, Leaving Certificate - Strand 1, Statistics and Probability**.

A9:3 **Workshop 1, Probability and Statistics**.

A9:4 **Student's CD**.

A9:5 **Teachers' Handbook for Junior Certificate, Strand 1, Probability and Statistics**.

A9:6 **Teachers' Handbook for Leaving Certificate Ordinary Level (Strand 1) Probability and Statistics**.

The next 11 items are **Teaching and Learning Plans** and all apply to both Junior Certificate and Leaving Certificate.

A9:7 **Teaching and Learning Plans 1: Introduction to Probability**.

A9:8 **Teaching and Learning Plans 2: Probability and Relative Frequency**.

A9:9 **Teaching and Learning Plans 3: Fair Trials with Two Dice**.

A9:10 **Teaching and Learning Plans 4: Outcome of Coin Tosses**.

A9:11 **Teaching and Learning Plans 5: Introduction to Playing Cards**.

A9:12 **Interactive Booklet to accompany Student's CD: Junior Certificate, Strand 1**.

A9:13 **Interactive Booklet to accompany Student's CD: Leaving Certificate, Strand 1**.

A9:14 **'Autograph' software file 'autograph.intro.file.pdf'**.

A9:15 **'Autograph' software file 'autograph.S2.pdf'**.

A9:16 **Excel 2003 in Mathematics teaching**.

A9:17 **Excel 2007 in Mathematics teaching**.

The following items are **Supplementary Material**.

A9:18 **Data Handling Cycle.pdf**.

A9:19 **Quick Guide on how to register Census at school**.

A9:20 **How to use Census at School**.

A9:21 **Student Activity: Fundamental Principle of Counting**.

A9:22 **Powerpoint presentation showing Fundamental Principle of Counting**.

For Strand 2 we list the following.

A9:23 **Mathematics Resources for Students** That for Junior Certificate states at the start of its *Introduction* that: *This booklet is designed to supplement the work you have done in Junior Cert geometry with your teacher. There are activities included for use as home work or in school.* Thus it consists of exercises in Geometry but it needs revision and corrections.

There is a similar booklet for Leaving Certificate.

A9:24 **Focus Workshop 2**

This has no introduction and no subdivisions. It contains activities for the pupils to perform, one of which is to be used in connection on the Student's CD. It deals with material for Junior and Leaving Certificates.

A9:25 **Student's CD**

This is a compact disk to be used with a computer. It exhibits various activities and has lists of exercises which can be printed off and used to pose questions to be dealt with in the active parts. It deals with material for Junior and Leaving Certificates.

A9:26 **Teacher Handbook; Common Introductory Course; Junior and Leaving Certificates; Synthetic Geometry; 'Adventures in Euclidean Geometry'** The purpose is to lay out teachers' plans lesson by lesson. There is no teaching material but that is laid out explicitly and systematically in Strand 2 Appendix 1. A special symbol is used to indicate that at the corresponding position of the content an interactive ICT module is available on the Student's CD. There are also many relevant references to files on the Project Math website.

A9:27 **Teacher Handbook; Geometry and Trigonometry; Junior Certificate** The purpose is to lay out teachers' plans lesson by lesson. There are four sections 1: Introduction, 2: Synthetic Geometry, 3: Coordinate Geometry, 4: Trigonometry. Section 2 just refers to [A9:26]. In Section 3, the items to be covered are listed but are not encountered in any presentation of the basic material. A similar remark applies to Section 4.

A9:28 **Teaching and Learning Plans; Plan 6: Planes and Points** This is for Junior Certificate.

A9:29 **Teaching and Learning Plans; Plan 7: Introduction to Angles** This is for Junior Certificate.

A9:30 **Teaching and Learning Plans; Plan 8: Introduction to Trigonometry** This is for Junior Certificate.

A9:31 **Teaching and Learning Plans; Using Pythagoras' theorem to establish the distance formula** This is for Junior Certificate. The title explains the contents. It has Student Activity sheets in common with Focus Workshop 2.

A9:32 **Teaching and Learning Plans; Plan 9: The Unit Circle** This is for Leaving Certificate.

A9:33 **Teaching and Learning Plans; Plan 10: Trigonometric Functions** This is for Leaving Certificate.

A9:34 **GeoGebra Overview; GeoGebra Sample Constructions; GeoGebra Co-ordinate Geometry of the Line; GeoGebra Co-ordinate Geometry of the Circle; GeoGebra Trigonometric Functions** These contain instructions for using the free graphics package GeoGebra on these items. These are prepared by the Project Maths Support Team.

A9:35 **Java Applets; Junior Certificate Geometry Applets; An Introduction to GeoGebra; Samples of Applet Construction using GeoGebra; A Companion Booklet for Java Applets created with Geogebra**. These have been prepared at the *Regional Centre for Excellence in Mathematics Teaching and Learning/University of Limerick*

The following is a lonely precursor for Strand 3.

A9:36 **Workshop 3 - Focus on Problem Solving & Strand 3, Number**. This is reproduced at end of *Appendix 5*.

The following is the report on the examination trialling in the twenty-four Pilot Schools.

A9:37 **Report on the Trialling of Leaving Certificate Sample Papers for Phase 1 of Project Maths**. State Examinations commission (January 2010).

Appendix 10
Determination of the Line of Regression

We outline here how this can be done for a finite set of points $(x_i,\ y_i,\ ),\ i = 1, 2, \ldots, n$ when they don't all lie on a vertical line.

The problem is to determine a pair of numbers m, c that minimise the sum of squares

$$f(m,c) = \sum_{i=1}^{n}(y_i - mx_i - c)^2,$$

when m, c are allowed vary. To handle this, first translate the data points to their centroid $(\bar{x}, \bar{y})$, via the transformation $X_i = x_i - \bar{x},\ Y_i = y_i - \bar{y}$, where

$$\bar{x} = \frac{1}{n}\sum_{k=1}^{n} x_k,\ \ \bar{y} = \frac{1}{n}\sum_{k=1}^{n} y_k.$$

Also, let $C = \bar{y} - m\bar{x} - c$, and then note that

$$y_i - mx_i - c = Y_i - mX_i - C,\ \ i = 1, 2, \ldots, n,$$

so that

$$\begin{aligned}
f(m,c) &= \sum_{i=1}^{n}(Y_i - mX_i - C)^2 \\
&= \sum_{i=1}^{n}\left((Y_i - mX_i)^2 - 2C(Y_i - mX_i) + C^2\right) \\
&= \sum_{i=1}^{n}(Y_i - mX_i)^2 + nC^2 \ \ (\text{since } \textstyle\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} Y_i = 0) \\
&= \left(m^2\sum_{i=1}^{n} X_i^2 - 2m\sum_{i=1}^{n} X_iY_i + \sum_{i=1}^{n} Y_i^2\right) + nC^2,
\end{aligned}$$

a sum of two simple quadratics, one in the variable m, and other in C. The latter is least when $\bar{y} - m\bar{x} - c = C = 0$, and the former when m minimises the nonnegative quadratic

$$m^2\sum_{i=1}^{n} X_i^2 - 2m\sum_{i=1}^{n} X_iY_i + \sum_{i=1}^{n} Y_i^2,$$

i.e., when

$$m = \frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} X_i^2},$$

this being well-defined since $\sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 > 0$ by our assumption. These values of m and c are unique, and determine, by definition, *the* Line of Regression with equation $y = mx + c$.

It should be noted, in particular, that the Line of Regression passes through the centroid of the data points.

While some Higher Level students will accept the above for general n, most of them should be able to deal with small values of n such as $n = 3, 4$, at least.

Again, formulae for the values of m, c could be made available in the Tables—assuming they are not there already—and Ordinary Level students, in particular, could be expected to use them to calculate the equation of the Line of Regression. Anything would be better than asking students to "draw the line of best fit by eye".

Appendix 11 Structure of the Syllabus

Traditionally syllabuses consisted mainly of headlines and were not very detailed. By contrast in Project Maths each Strand has what may be referred to as a *Standards* specification which seeks to control in detail what standards students are expected to achieve.

There is an essentially uniform format for the syllabuses of the five Strands but there is one difference between those for Junior Certificate on the one hand and those for Leaving Certificate on the other.

Each of the Syllabuses for both the Junior Certificate and the Leaving Certificate contain in their introductory material a subsection of the section **Syllabus overview** entitled **Teaching and learning**. These subsections are thoroughly based on the context-constructivist approach, which is the approach to be found in PISA and intended to be pervasive.

Structure of syllabuses for Junior Certificate

At the start of each Strand there is a list stating what students will do. In the case of Strand 1 it is

• use ..., • explore ..., • develop ..., • complete ...,

and in the case of Strand 2 it is

• recall ..., • construct ..., • solve ...... logical proofs ...,

• analyse and process ...,• select ... and apply ....

Then there is the actual formal syllabus, page by page, in the format

Topic	Description of topic Students learn about	Learning outcomes Students should be able to

Structure of syllabuses for Leaving Certificate

In the introductory material for this section there is in addition a subsection entitled **Key skills** which are listed as *information processing, being personally effective, communicating, critical and creative thinking* and *working with others*, the provenance of which is also PISA.

At the beginning of each Strand for Leaving Certificate there is brief section giving the aims and approach for the Strand. In the case of Strand 1 there is a paragraph on Probability, as providing certain understandings intrinsic to problem solving and it underpins the Statistics unit. There is also a paragraph on Statistics, identifying problems to be explored by use of data, designing, collecting, exploring, solving, communicating, interpreting, evaluating, dealing with uncertainty and variation.

For Strand 2, which embraces *Synthetic geometry, Coordinate geometry, Trigonometry, Transformation geometry, Problem solving and synthesis skills*, the note seems to be largely confined to Synthetic geometry.

Then there is the actual formal syllabus, page by page, in the format

Students learn about	Students working at FL should be able to	In addition, students working at OL should be able to	In addition students working at HL should be able to

Boldface is used just to identify constructions, theorems and corollaries by number.

Appendix 12 Transformation Geometry

Summary
A prominent constructivist system is one which is produced by the National Council of Teachers of Mathematics (NCTM) in the U.S.A. It is very detailed and comprehensive giving a good understanding of what this type of approach can entail. We look at just one topic in it, namely *Geometrical Transformations*, and go on to cast a critical eye on the corresponding section in Project Math, which does not contain a statement of even one general property.

1. A prominent constructivist system is that which is based on the following National Council of Teachers of Mathematics (USA) publications

   (a) 2000 Principles and Standards for School Mathematics,

   (b) 2006 Curriculum Focal Points.

   The material is divided into the following sections

   Pre-K-Grade 2, Grades 3-5, Grades 6-8, Grades 9-12.

   Thus Grades 6-8 represent Junior High School and Grades 9-12 represent Senior High School.

2. While we are not supportive of this constructivist system, we find very valuable in these publications the tables of standards which are followed by extensive explanatory material giving thorough comments on objectives, principles, reasons, various approaches taken, and the many examples included. It comprehensively conveys a detailed understanding of one major constructivist system.

3. Taking up the topic of geometrical transformations in the NCTM publications, it is pointed out there that the application of informal transformations such as flips, turns, slides and scaling, together with the use of symmetry, can be used to describe the sizes, positions and orientations of shapes. Progressing to the set of transformations consisting of translations, reflections, axial symmetries, and rotations an understanding of congruence of triangles is developed.

So, for example, given a triangle $\Delta(A, B, C)$ subjected to one of these transformations, the image will be a triangle $\Delta(A', B', C')$ congruent to the first, and we can choose the notation so that $|\angle A| = |\angle A'|, |\angle B| = |\angle B'|, |\angle C| = |\angle C'|$.

That is straight-forward and is the easy part, but working in reverse we can take congruent triangles $\Delta(A, B, C)$ and $\Delta(A', B', C')$ and ask if there is one of these transformations, or a composition of two or more of them, which maps the first of these triangles onto the second of them. Geometers know how to do this, and to cover all cases all of the above types of transformation are needed.

However to deal with similarity of triangles as well, the use of *enlargements* or *dilations* needs to be involved as well as all the ones above.

4. (a) If we now look at transformation geometry for Junior Certificate in Project Maths, we have the formal transformations *translations, axial symmetries* and *central symmetries* and no purpose or objective is given for them. We have not got general rotations as the only one is central symmetry which is equivalent to rotation through 180°. Thus they cannot deal fully with congruent triangles except by representing rotations as combinations of axial symmetries in a complicated way.

   For Leaving Certificate in Project Maths there is mention only of enlargements. While these are useful for scaling figures, no effort is made to deal with similar triangles in general.

   (b) One very commonly used phrase in Project Maths geometry with which we have difficulty is *"the orientation"*. But orientation is not defined.

   (c) There does not seem to be a clear prominent statement in Project Maths that geometry is an abstract mathematical model of space. Diagrams are approximate, so are constructions, so are lengths, so are angle measures, even when done on computers as these operate and are rounded off only to so many decimal places.

Recommendation We recommend that in Project Maths the section on Transformation Geometry either be upgraded as suggested as in *Appendix 1*, or else be omitted entirely.

Appendix 13 Further Comments

1. **Terminology**

When words such as *realistic*, *real world*, *real life* and *problem-solving* are used in Project Maths then the meanings are not necessarily what is normally understood in ordinary usage. There is then a danger of misinterpretation by the majority of readers who are not specialists in Mathematics Education.

(a) For example, the word *realistic*, which is used very much in RME, and the words *real world* and *real life* which are used in PISA and Project Maths, stand for the same concept.

(b) The PISA system or programme is named *Mathematical Literacy* rather than Mathematics and to show what its authors mean by this term we refer to Chapter 3, entitled *Mathematical Literacy*, of [35]. Discussing the problem of assessing whether 15-year old students are mathematically literate in terms of their ability to mathematise, it is pointed out that this is difficult in a timed assessment because in " *most complex real situations the process of proceeding from reality to Mathematics involves collaboration and finding appropriate resources*". Judging whether such students can use their accumulated mathematical knowledge to solve mathematical problems encountered in their world, one should collect information about their ability to mathematise complex situations. This is impractical, so PISA chooses items to assess different parts of the process. A strategy is adopted which creates "*a set of test items in a balanced manner so that a selection of these items cover the five aspects of mathematising. The aim is to use the responses to those items to locate students on a scale of proficiency in the PISA construct of mathematical literacy*".

So in general, PISA puts an emphasis on tasks that "*might be encountered in some real-world situation and possess an authentic context for the use of Mathematics that influences the solution and its interpretation*". The authors continue by pointing out that "*this does not preclude the inclusion of tasks in which the context is hypothetical, as long as the context has some real elements, is* *not too far removed from a real-world situation, and for which the use of Mathematics to solve the problem would be authentic."*

It should be noted that PISA uses the term " authentic" to indicate that the use of Mathematics is "*genuinely directed to solving the problem at hand, rather than the problem being merely a vehicle for the purpose of practising some Mathematics."*

(c) One feature of both RME and PISA which receives great attention is their claim to focus on and develop *higher order skills* in contrast with more traditional approaches which deal only with *lower order skills* and thus that they are providing a deeper understanding of Mathematics.

These skill classifications became a major educational item with the 1956 publication of Bloom's taxonomy of educational objectives [37]. This defined the higher order thinking skills, in descending order, as *creating, evaluating, analysing*, and the lower order thinking skills, in ascending order, as *remembering, understanding, applying.* With the growth of later approaches to Mathematics education, the proponents of these added further concepts to the list, on the one hand, of *higher order* thinking skills, such as discovery, open-ended problems, using Mathematics in the real world, providing reasons to support conclusions, procedures such as integrating and analysing and solving problems, and, on the other hand, to the list of *lower order* thinking skills were added algorithms, manipulating symbols, abstraction, remembering formulae and procedures, understanding and recalling and using strategies and implementing procedures.

In PISA, there are six levels in the testing of *competencies* and there is a table in [7] and [8] listing the details of the six levels, and frequent references to *relational understanding* and *instrumental understanding*, and to *lower-level competencies* and *higher-level competencies.*

Comment We note a reproachful reference to *symbol manipulation* in the above classification of skills and feel compelled to reply. First of all there are many symbols in arithmetic e.g. numerals (e.g. 1,2,3), signs for operations $2+3, 2-3, 2\times 3, 2\div 3, \sqrt{4}$, signs for relations $2=\sqrt{4}, 2<3, 3>2$, and these are manipulated a lot.

The use of symbols in fact makes Mathematics much easier to do and communicate. It enables us to do more and better Mathematics. Why should this lead it to be disparaged as a lower order thinking skill? Mathematicians do not use prose word-processing for their work; they use special mathematical word-processing to handle the many symbols. If there was a bar to the intelligent use of symbols in it, Mathematics would be gutted.

RME advocates are trying to confine Mathematics to what can be dealt with by arithmetic and seem unappreciative of the remarkable progress in Mathematics since 1400A.D.

2. **Remarks concerning the Second Level Curriculum prior to 2000.**

We quote from [1,p.11]: "*The style of the Present Leaving Certificate syllabus was set in the 1960s at the time of the 'modern Mathematics revolution'. This emphasized abstraction, rigorous argument and use of precise terminology.*" Continuing, the authors say "*The 'modern' emphasis has been diluted in subsequent revisions, and a more eclectic philosophy has taken its place.*"

Statements like the first part of this quotation, without the disarming second part, are common in Irish articles on Mathematics Education. For example, several instances can be found thoughout [2].

We dispute this characterization of the pre-2000 syllabus and believe that it needs to be seriously challenged.

3. **Tendency of the Project Maths documentation**

We detect a pro-RME and pro-PISA slant in the documentation published or commissioned over the last decade by the NCCA, the Department of Education and Skills, Project Maths, the Educational Research Centre, Dublin, and many articles on Mathematics Education published in Ireland. We give two examples.

(a) Perhaps the most explicit instance of this is in [1,p.6] where, referring to RME, we find "*It is probably the most 'fashionable' approach among Mathematics educators at present and underpins the OECD Programme for Student Assessment* (PISA)".

(b) In the consultation document *Review of Mathematics in Post-primary education: Consultation Questionnaire, 2005*, on page 3, question 4 asks for comment on the relative merits of the following approaches to the Junior Certificate and Leaving Certificate Mathematics courses

• *'modern Mathematics'*[,] *with its emphasis on abstraction, logical structure, rigorous argument, set theory, number theory, etc.*

• *real-world or context-based Mathematics, also referred to as 're-alistic Mathematics education* (RME)'.

We believe that this is a loaded question as the first option is bizarre.

Part IV

References

All the files referred to here can be located via GOOGLE.

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For more lists of problems.

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33 EDinformatics, TIMSS Sample, Middle School, Mathematics Test (Grades7 and 8) http://www.edinformatics.com/timss/pop2/mpop2.htm?submit324=Grade+7

For specific material on PISA, see [4] and

34 *The PISA 2003 Assessment Framework: Mathematics, reading, science and problem solving knowledge and skills. OECD. Paris 2003*

35 *Assessing Scientific, Reading and Mathematical Literacy: A Framework for PISA 2006, OECD Paris 2006*, Chapter 3, pp.71-117 entitled *Mathematical Literacy*

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