Rotations in Plane Geometry
Group Project 3 for Project Maths, Strand 2, Higher Levels
by
Patrick D. Barry
University College, Cork.

From our intuition in the physical world around us of rotation about an axis as a disciplined form of twisting, in plane geometry we have the concept of a plane Π rotating rigidly about a point O by an amount up to a complete revolution. Thus the physical concept of a wheel twisting about a fixed axis by an amount up to a complete turn conveys the geometrical analogue more closely than does that of either a moving hand on a clock or of a wheel spinning on an axle.

1 Rotations

A rotation is a function from the plane Π onto the plane Π with specific properties. We first define a rotation about a point O through an angle $\angle AOB$. This is done by specifying the image P' of each point P in the plane under the rotation. We go on to prove specific properties.

1.1 Preparatory material; orientation

For distinct points $P_2 = (x_2, y_2),\ P_3 = (x_3, y_3)$, the well-known equation

$$y - x_2 = \frac{y_3 - y_2}{x_3 - x_2}(x - x_2)$$

covers all lines which are not parallel to the y-axis. The slightly modified form of this

$$(x - x_2)(y_3 - y_2) - (y - y_2)(x_3 - x_2) = 0$$

covers all lines in the plane. It follows that

$$(x - x_2)(y_3 - y_2) - (y - y_2)(x_3 - x_2) \neq 0$$

covers all cases where P, P_2, P_3 are not collinear, that is when PP_2P_3 is a triangle.

We select a base triangle OIJ, where $O = (0,0),\ I = (1,0),\ J = (0,1)$ and say that the circuit $O \to I \to J \to O$ around the boundary of this triangle is *anticlockwise*. We wish to transfer this concept to arbitrary triangles $P_1P_2P_3$. For this, given a triangle $P_1P_2P_3$, where $P_1 = (x_1, y_1)$, we seek a transformation

$$\begin{aligned} x' =& ax + by + k_1, \\ y' =& cx + dy + k_2, \end{aligned}$$

under which we have $O \to P_1,\ I \to P_2,\ J \to P_3$.

For $(0, 0) \to (x_1, y_1)$ we have

$$x_1 = a.0 + b.0 + k_1 = k_1, y_1 = c.0 + d.0 + k_2 = k_2.$$

For $(1, 0) \to (x_2, y_2)$ we have

$$x_2 = a + b.0 + k_1 = a + k_1, y_2 = c + d.0 + k_2 = c + k_2.$$

For $(0, 1) \to (x_3, y_3)$ we have

$$x_3 = a.0 + b + k_1 = b + k_1, y_3 = c.0 + d + k_2 = d + k_2.$$

From these we have that

$$\begin{aligned} k_1 =& x_1, k_2 = y_1, ; a = x_2 - k_1 = x_2 - x_1, ; c = y_2 - k_2 = y_2 - y_1, \\ b =& x_3 - k_1 = x_3 - x_1, ; d = y_3 - k_2 = y_3 - y_1. \end{aligned}$$

Thus we have the transformation

$$\begin{aligned} x' - x_1 =& (x_2 - x_1)x + (x_3 - x_1)y, \\ y' - y_1 =& (y_2 - y_1)x + (y_3 - y_1)y. \end{aligned}$$

As from the coefficients of this we have

$$D = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1),$$

we must have $D \neq 0$ as $P_1P_2P_3$ is a triangle.

When $D > 0$ we say that the order $P_1 \to P_2 \to P_3 \to P_1$ of a circuit of the boundary of the triangle $P_1P_2P_3$ is *anticlockwise*. When $D < 0$ we say that the order $P_1 \to P_2 \to P_3 \to P_1$ of a circuit of the boundary of the triangle $P_1P_2P_3$ is *clockwise*.

If we interchange P_2 and P_3 we obtain

$$D' = (x_3 - x_1)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_1) = -D,$$

and so this interchanges the orientations into anticlockwise or clockwise.

Example We take non-collinear points $P_1 = (x_1, 0), P_2 = (x_2, 0),\ \ P_3 = (x_3, y_3)$ where $x_1 < x_2$ and $y_3 > 0$. Then P_1 and P_2 lie on the x-axis with P_1 to the left of P_2 and P_3 is any point in the upper half-plane. Then

$$D = (x_2 - x_1)(y_3 - 0) - (x_3 - x_1)(0 - 0) = (x_2 - x_1)y_3 > 0.$$

Thus the circuit $P_1 \to P_2 \to P_3 \to P_1$ is anticlockwise.

1.2 Definition of a rotation

We start with a point O and an angle $\angle AOB$ such that the circuit $O \to A \to B \to O$ of the boundary of the triangle OAB is anticlockwise. We refer to such an angle as *anticlockwise*. We wish to define rotation about the point O through the angle $\angle AOB$.

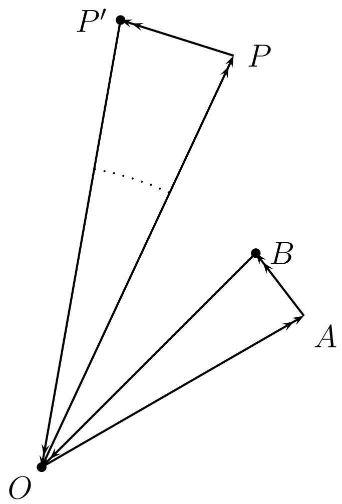

Figure 1.

Definition:-

A rotation about a point O through an anticlockwise angle $\angle AOB$, is a function from Π onto Π such that:-

(i) For each point $P \neq O$, the image P' of P is the point such that $|OP'| = |OP|$ and the angle $\angle POP'$ is such that the circuit $O \to P \to P' \to O$ of the boundary of the triangle OPP' is anticlockwise, and $|\angle POP'| = |\angle AOB|$.

(ii) For $P = O, P' = O' = O.$

1.3 Images of a pair of points

We look at a pair of points P and Q and their images P' and Q'.

1.3.1 First case

We take a case as shown in Figure 2.

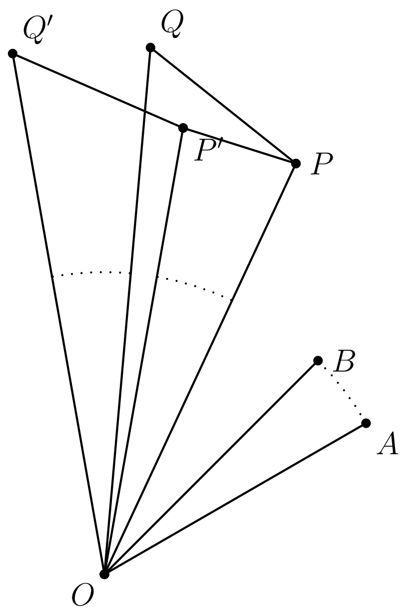

Figure 2.

We draw the segments $[P, Q]$ and $[P', Q']$. We note that the triangles OPQ and $OP'Q'$ are congruent as $|OP| = |OP'|$, $|OQ| = |OQ'|$, and

$$\begin{aligned} |\angle POQ| =& |\angle POP'| + |\angle P'OQ| \\ =& |\angle QOQ'| + |\angle P'OQ| \\ =& |\angle P'OQ'|. \end{aligned}$$

The important conclusion we take from this is that $|PQ| = |P'Q'|$, i.e. distance is preserved.

1.3.2 Second case

We take a case as shown in Figure 3.

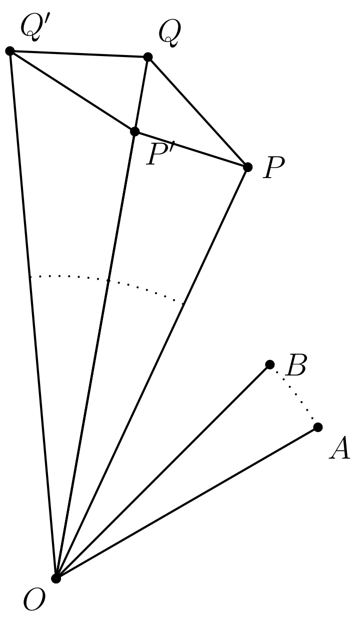

Figure 3.

It is clear that the triangles OPQ and $OP'Q'$ are congruent again and we deduce that $|PQ| = |P'Q'|$.

1.3.3 Third case

We take a case as is shown in Figure 4.

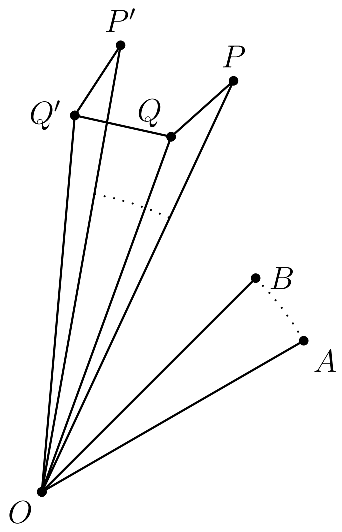

Figure 4.

We again have that the triangles OPQ and $OP'Q'$ are congruent. For we have that

$$\begin{aligned}|\angle POQ| =&|\angle POP'| - |\angle QOP'| \\ =&|\angle QOQ'| - |\angle QOP'| \\ =&|\angle P'OQ'|.\end{aligned}$$

Again we reach the earlier conclusions.

1.4 Images of three distinct collinear points P, Q and R

We build our diagrams on Figure 2.

1.4.1 The case of $R \in [P, Q]$

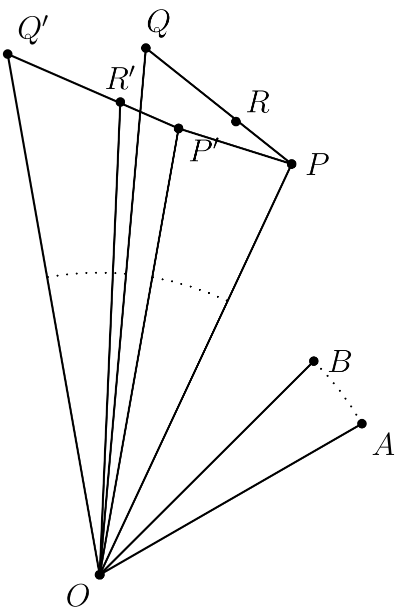

Figure 5.

As $R \in [P, Q]$ we have $|PR| + |RQ| = |PQ|$. It follows that $|P'R'| + |R'Q'| = |P'Q'|$ and hence $R' \in [P', Q']$.

Conversely let $T \in [P', Q']$. Then $|P'T| < |P'Q'| = |PQ|$. Now take the point $S \in [P, Q]$ such that $|PS| = |P'T|$. It follows that $T = S'$. Hence the segment $[P, Q]$ is mapped onto the segment $[P', Q']$.

1.4.2 The case of $Q \in [P, R]$

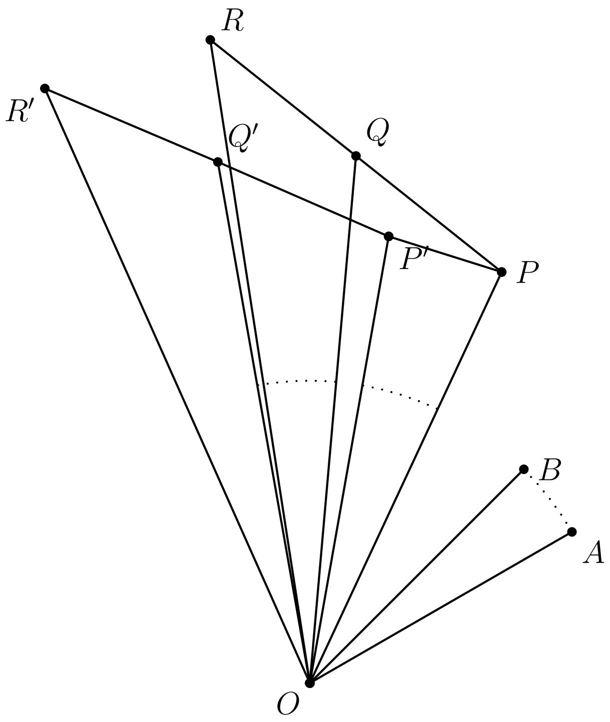

Figure 6.

This time we start with R in the half-line $[P,\ Q$ but beyond Q in it. By the last section we have that $|P'R'| = |PR|$ and $Q' \in [P', R']$. Thus the half-line $[P,\ Q$ is rotated into the half-line $[P',\ Q'$, initially the segment $[P, Q]$ onto $[P', Q']$ from the last subsection and now the points R beyond Q onto the points R' beyond Q'. Conversely if we take a point T on the line $P'Q'$ and beyond Q' we will have $|P'T| > |P'Q'| = |PQ|$ and if we take the point S on the half-line $[P,\ Q$ so that $|PS| = |PT|$ we will have that $T = S'$. Thus the half-line $[P,\ Q$ rotates onto the half-line $[P',\ Q'$.

This argument also applies to the half-line $[Q,\ P$ mapping onto the half-line $[Q',\ P'$.

1.4.3 The case of the whole line PQ

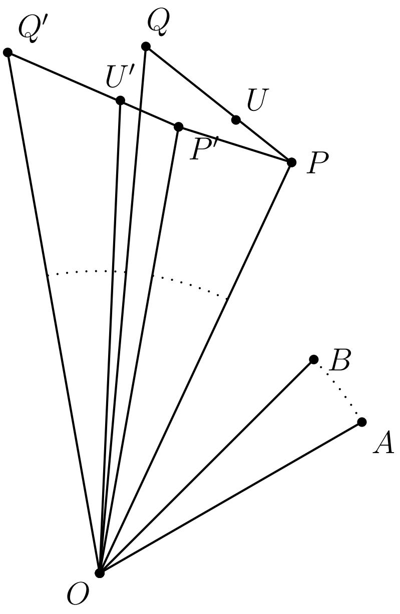

Figure 7.

Now on the line PQ take a distinct point U lying between P and Q on the line. Then $PQ = [U, \ Q \cup [U, \ P$ and these half-lines have only the point U in common. Then U' lies on the line $P'Q'$ between P' and Q'. By the last subsection $[U, \ Q$ is rotated onto $[U', \ Q'$ and $[U, \ P$ is rotated onto $[U', \ P'$. On taking the union of the half-lines we see that PQ is rotated onto $P'Q'$.

Exercise Starting with a segment $[O, Q]$ instead of $[P, Q]$, prove analogues of §1.2 and §1.3.

2 Composition of rotations

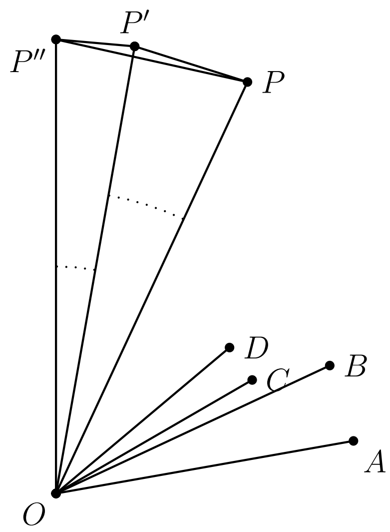

Figure 8.

With the one centre O we wish to consider rotation about the point O through the anticlockwise angle $\angle AOB$ followed by rotation about the point O through the anticlockwise angle $\angle COD$. Then the composition of $P \to P'$ followed by $P' \to P''$ yields $P \to P''$, and

$$|\angle POP'| + |\angle P'OP''| = |\angle POP''|.$$

3 Clockwise rotations

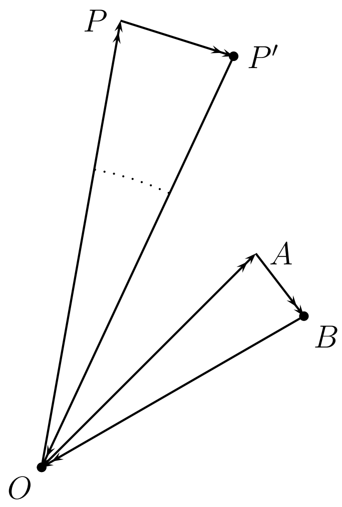

Figure 9.

For clockwise rotations about the point O we modify the foregoing material as follows. We take an angle $\angle AOB$ such that the circuit $O \to A \to B \to O$ of the boundary of the triangle OAB is clockwise. We refer to such an angle as *clockwise*. We wish to define rotation about the point O through the angle $\angle AOB$.

Definition:-

A rotation about a point O through a clockwise angle $\angle AOB$, is a function from Π onto Π such that:-

(i) For each point $P \neq O$, the image P' of P is the point such $|OP'| = |OP|$ and the angle $\angle POP'$ is such that the circuit $O \to P \to P' \to O$ of the boundary of the triangle OPP' is clockwise, and $|\angle POP'| = |\angle AOB|$.

(ii) For $P = O, P' = O' = O$.

For properties of clockwise rotations we take direct analogues of the properties of anticlockwise rotations laid our in §1 and §2

4 Trigonometric formulae for anticlockwise rotations

4.1 Formulae for the cosine and sine of the difference of two angles

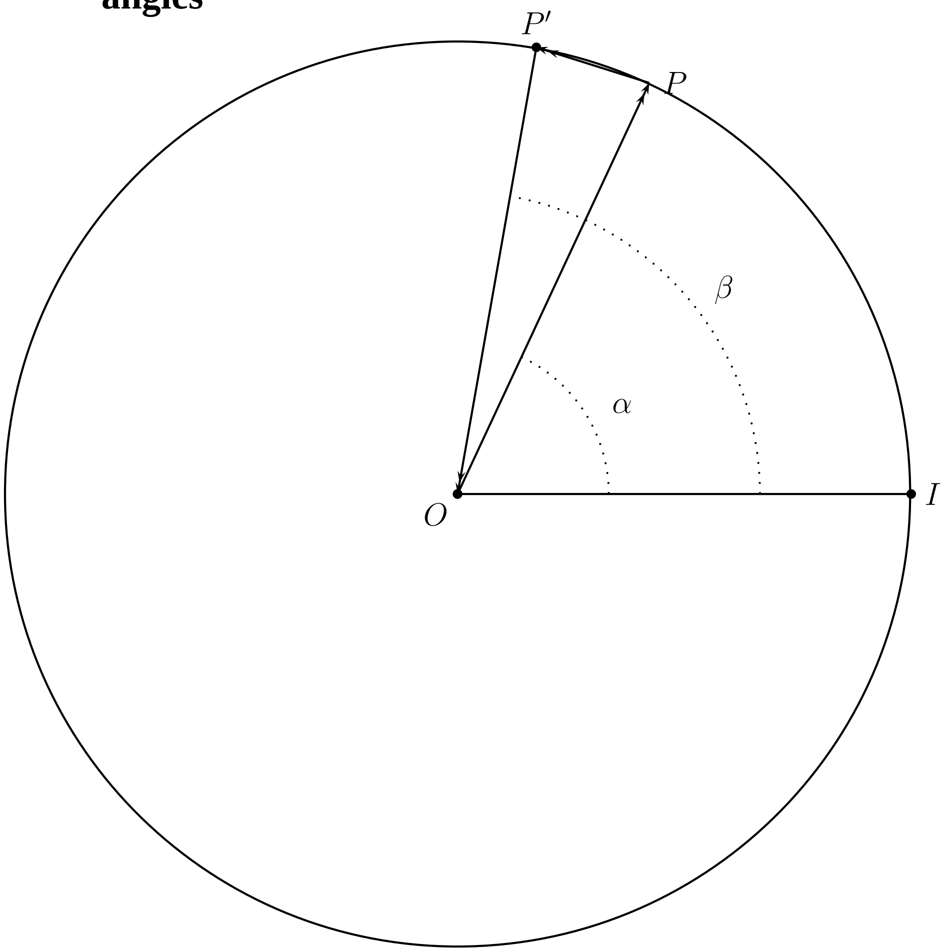

Figure 10.

We take a circle with centre O and radius length r. Let I be the point with coordinates $(r, 0)$. Let the point P on the circle have coordinates (x, y) and denote the anticlockwise angle $\angle IOP$ by α. Let the point P' on the circle have coordinates (x', y') and denote the anticlockwise angle $\angle IOP'$ by β, and suppose that $|\alpha| < |\beta|$.

Then by the cosine rule we have that

$$|PP'|^2 = |OP|^2 + |OP'|^2 - 2|OP| \cdot |OP'| \cos(\beta - \alpha),$$

so that

$$\begin{aligned}
(x - x')^2 + (y - y')^2 =& 2r^2 - 2r^2 \cos(\beta - \alpha), \\
2r^2 - 2(xx' + yy') =& 2r^2 - 2r^2 \cos(\beta - \alpha), \\
r^2 \cos(\beta - \alpha) =& xx' + yy', \\
\cos(\beta - \alpha) =& \frac{xx' + yy'}{r^2}, \\
\cos(\beta - \alpha) =& \frac{x}{r}\frac{x'}{r} + \frac{y}{r}\frac{y'}{r} \\
\cos(\beta - \alpha) =& \cos\alpha\cos\beta + \sin\alpha\sin\beta.
\end{aligned}$$

Now

$$\sin^2(\beta - \alpha) = 1 - \cos^2(\beta - \alpha) = 1 - \frac{(xx' + yy')^2}{r^4},$$

so that

$$\begin{aligned}
r^4 \sin^2(\beta - \alpha) =& r^4 - (xx' + yy')^2 \\
=& (x^2 + y^2)(x'^2 + y'^2) - (xx' + yy')^2 \\
=& (xy' - x'y)^2.
\end{aligned}$$

Now for the circuit $O \to P \to P' \to O$ of the boundary of the triangle OPP' the expression D in §1.1 is $xy' - x'y$ and this is positive. Hence we have that

$$\sin(\beta - \alpha) = \frac{xy' - x'y}{r^2}$$

and thus we have that

$$\begin{aligned}
\sin(\beta - \alpha) =& \frac{x}{r}\frac{y'}{r} - \frac{x'}{r}\frac{y}{r} \\
=& \sin\beta\cos\alpha - \cos\beta\sin\alpha.
\end{aligned}$$

The usual proof of the cosine rule involves an angle of a triangle which must have a magnitude of less than $180°$, so that in our notation $|\beta - \alpha| < 180$. We cope with extending this to reflex angles as in the following diagram. We note that we deal with the anticlockwise angle $\angle P'OP$ and that $|\angle P'OP| = 360 - |\angle POP'|$, and hence $\cos\angle P'OP = \cos\angle POP'$. Thus the cosine rule extends to this case.

Exercise. Prove the cosine rule for the case where $|OP| = |OP'| = r$ and $|\angle POP'| = 180$.

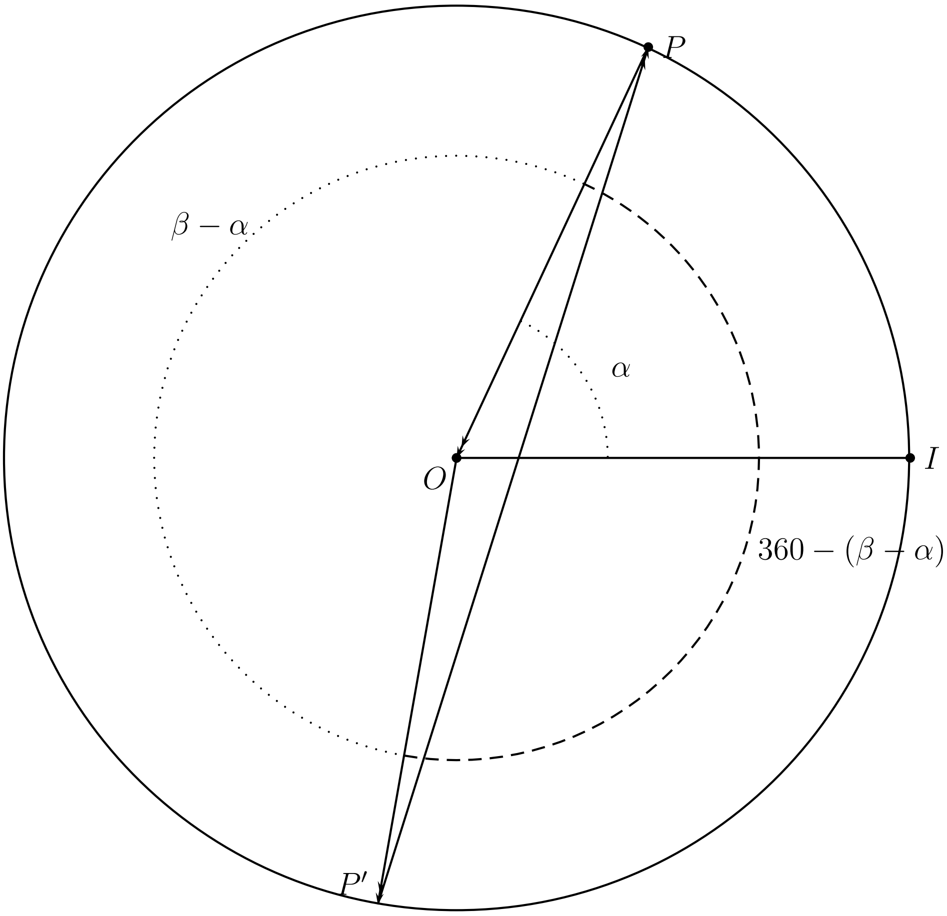

Figure 11.

4.2 Formulae for the cosine and sine of a double angle

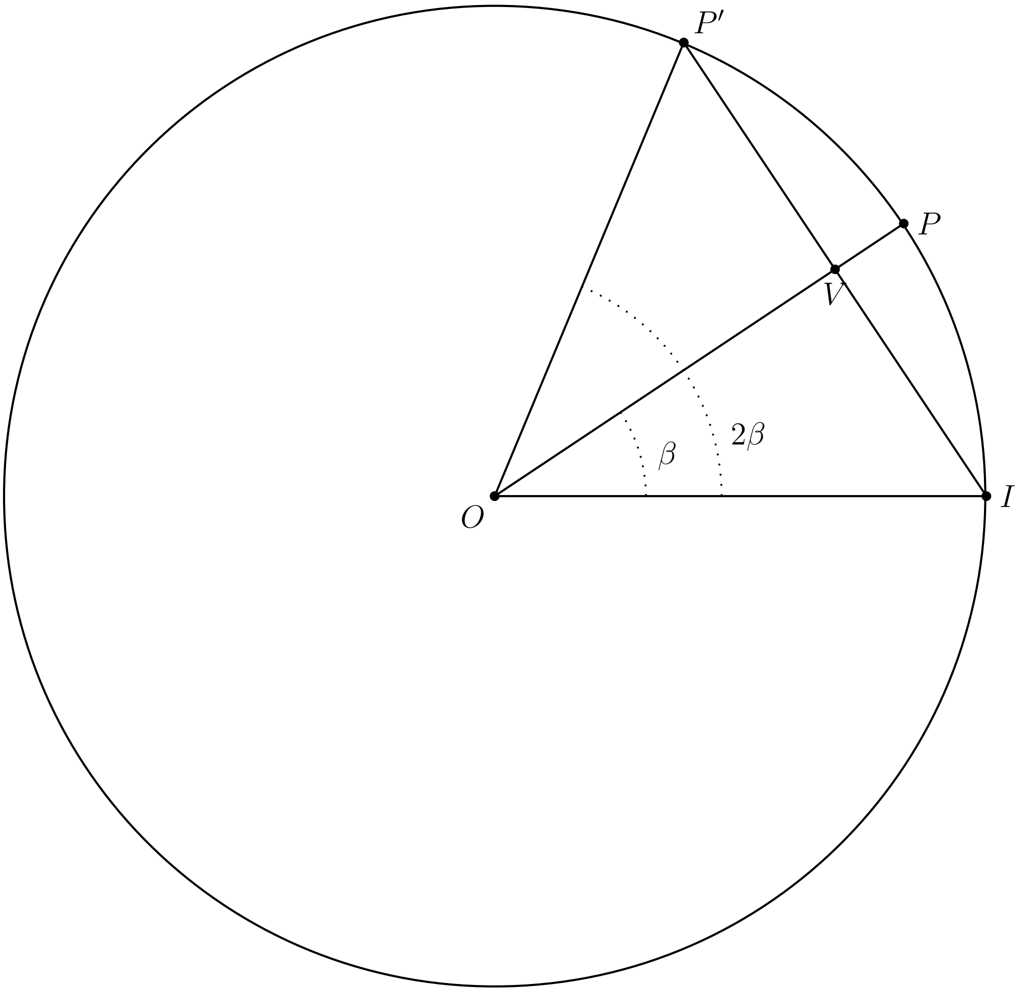

Figure 12.

We take a circle with centre O and radius length r. Let I be the point with coordinates $(r, 0)$. Let the point P on the circle have coordinates (x, y) and denote the anticlockwise angle $\angle IOP$ by β. Let the point P' on the circle have coordinates (x', y') and denote the anticlockwise angle $\angle IOP'$ by 2β.

Now the midpoint V of I and P' has coordinates

$$V = (\frac{r\cos 2\beta + r}{2}, \frac{r\sin 2\beta}{2}) = \frac{r}{2}(1 + \cos 2\beta, \sin 2\beta).$$

Now OIP' is an isosceles triangle and so $OV \perp IP'$. Then

$$\begin{aligned}|OV|^2 =& \frac{r^2}{4}\left((1+\cos 2\beta)^2 + \sin^2 2\beta\right)\\ =& \frac{r^2}{4}\left(1 + 2\cos 2\beta + \cos^2 2\beta + \sin^2 2\beta\right)\\ =& r^2\frac{1+\cos 2\beta}{2}\\ |OV| =& r\sqrt{\frac{1+\cos 2\beta}{2}}.\end{aligned}$$

Then $\cos\beta = \frac{|OV|}{r} = \sqrt{\frac{1+\cos 2\beta}{2}}$.

Similarly

$|IP'|^2 = (r\cos 2\beta - r)^2 + r^2\sin^2 2\beta = r^2 - 2r^2\cos 2\beta + r^2 = 2r^2(1-\cos 2\beta)$,

and so $|IP'| = r\sqrt{2(1-\cos 2\beta)}$ and

$$\frac{|IV|}{r} = \frac{|IP'|}{2r} = \sqrt{\frac{1-\cos 2\beta}{2}}.$$

Then

$$\sin\beta = \frac{|IV|}{r} = \sqrt{\frac{1-\cos 2\beta}{2}}.$$

Now $\cos^2\beta = \frac{1}{2}(1+\cos 2\beta)$ so

$$\cos 2\beta = 2\cos^2\beta - 1 = 1 - 2\sin^2\beta = \cos^2\beta - \sin^2\beta.$$

Moreover

$$\begin{aligned}\cos\beta\sin\beta =& \sqrt{\frac{1-\cos^2 2\beta}{4}} = \sqrt{\frac{\sin^2 2\beta}{4}}\\ =& \pm\frac{1}{2}\sin 2\beta,\end{aligned}$$

and so

$$\sin 2\beta = \pm 2\sin\beta\cos\beta.$$

Now we are to take the $+$ sign here instead of $\pm$. To see this we denote the four quadrants by I, II, III and IV. Then if $\beta \in I$ we have $2\beta \in I$ or II. If $\beta \in II$ we have $2\beta \in III$ or IV. If $\beta \in III$ we have $2\beta \in I$ or II. If $\beta \in IV$ we have $2\beta \in III$ or IV. It can be checked that in each of these cases the $+$ sign is appropriate.

4.3 Formulae for the cosine and sine of the sum of two angles

When $|\alpha| < |\beta|$ we note that $2\beta - (\beta - \alpha) = \beta + \alpha$. Then

$$\begin{aligned}\cos(\beta+\alpha) &= \cos 2\beta \cos(\beta-\alpha) + \sin 2\beta \sin(\beta-\alpha)\\ &= \cos 2\beta[\cos\beta\cos\alpha + \sin\beta\sin\alpha] + \sin 2\beta[\sin\beta\cos\alpha - \cos\beta\sin\alpha]\\ &=[\cos 2\beta\cos\beta + \sin 2\beta\sin\beta]\cos\alpha + [\cos 2\beta\sin\beta - \sin 2\beta\cos\beta]\sin\alpha\\ &= \cos(2\beta-\beta)\cos\alpha - \sin(2\beta-\beta)\sin\alpha\\ &= \cos\beta\cos\alpha - \sin\beta\sin\alpha.\end{aligned}$$

Similarly

$$\begin{aligned}\sin(\beta+\alpha) &= \sin 2\beta \cos(\beta-\alpha) - \cos 2\beta \sin(\beta-\alpha)\\ &= \sin 2\beta[\cos\beta\cos\alpha + \sin\beta\sin\alpha] - \cos 2\beta[\sin\beta\cos\alpha - \cos\beta\sin\alpha]\\ &=[\sin 2\beta\cos\beta - \cos 2\beta\sin\beta]\cos\alpha + [\sin 2\beta\sin\beta + \cos 2\beta\cos\beta]\sin\alpha\\ &= \sin(2\beta-\beta)\cos\alpha + \cos(2\beta-\beta)\sin\alpha\\ &= \sin\beta\cos\alpha + \cos\beta\sin\alpha.\end{aligned}$$

5 Coordinate form of a rotation

With the notation of Figure 10 we denote $\beta - \alpha$ by γ. Then we have

$$x' = r\cos\beta,\ y' = r\sin\beta,\ x = r\cos\alpha,\ y = r\sin\alpha.$$

Then

$$\begin{aligned}r^2\cos\gamma &= r^2\cos(\beta-\alpha)\\ &= r^2\cos\beta\cos\alpha + r^2\sin\beta\sin\alpha\\ &= r\cos\beta . r\cos\alpha + r\sin\beta . r\sin\alpha\\ &= x'x + y'y.\end{aligned}$$

Similarly

$$\begin{aligned}r^2\sin\gamma &= r^2\sin(\beta-\alpha)\\ &= r\sin\beta\ r\cos\alpha - r\cos\beta\ r\sin\alpha\\ &= y'x - x'y.\end{aligned}$$

Then

$$\begin{aligned} xx' + yy' =& r^2 \cos\gamma, \\ -yx' + xy' =& r^2 \sin\gamma. \end{aligned}$$

When we solve these linear equations for x' and y', using the fact that $x^2+y^2 = r^2$, we obtain the solutions

$$x' = x\cos\gamma - y\sin\gamma, \quad y' = x\sin\gamma + y\cos\gamma.$$