Minimality and Comparison of Sets of Multi-Attribute Vectors

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Introduction

In a decision-making problem, outcomes or alternatives may only be partially ordered with respect to the available preference information, for instance in a multi-criteria or multi-objective problem. Suppose that, in a particular situation, $\boldsymbol{A}$ is the set of outcomes that are available to the decision maker. This is interpreted in a disjunctive fashion, in that the user is free to choose any element $\boldsymbol{\alpha}$ of $\boldsymbol{A}$. However, as is common, we do not know precisely the user's preferences. The preference information available to the system is represented in terms of a set $\boldsymbol{\mathcal{W}}$ of scenarios (i.e., user preference models) where, associated with each scenario $\boldsymbol{w} \in \boldsymbol{\mathcal{W}}$, is a (real-valued) utility function $\boldsymbol{f_w}$ over outcomes.

We consider, in particular, the following related pair of questions:

(1) **Minimality**: are there elements of $\boldsymbol{A}$ that can be eliminated unproblematically? In particular, is there a strict subset $\boldsymbol{A'}$ of $\boldsymbol{A}$ that is equivalent to $\boldsymbol{A}$?

(2) **Sets comparison**: Given a choice between one situation, in which the available outcomes are $\boldsymbol{A}$, and another situation, in which outcomes $\boldsymbol{B}$ are available, is $\boldsymbol{A}$ at least as good in every scenario?

User preferences

Each element $\boldsymbol{w}$ of $\boldsymbol{\mathcal{W}}$ is viewed as a possible model of the user's preferences that is consistent with the preference information we know. If we knew that $\boldsymbol{w}$ were the true scenario, so that $\boldsymbol{f_w}$ represents the user's preferences over outcomes, then we would be able to choose a best element of $\boldsymbol{A}$ with respect to $\boldsymbol{f_w}$ leading to a utility value $Val_{\boldsymbol{A}}(\boldsymbol{w}) = \max_{\alpha \in A} f_w(\alpha)$. Consider for example the set $\boldsymbol{A''}$ in Figure 1, if we knew that the user preference was $\boldsymbol{w} = (\frac{2}{3}, \frac{1}{3})$, then $Val_{\boldsymbol{A}}(\boldsymbol{w}) = \max(f_w(10,4), f_w(4,7)) = \max(8,5) = 8$ that is the utility value associated to the element $(4,7)$ of $\boldsymbol{A}$ in $\boldsymbol{w} = (\frac{2}{3}, \frac{1}{3})$. However, the situation can be ambiguous given a non-singleton set $\boldsymbol{\mathcal{W}}$ of possible user models or scenarios. Considering again the example in Figure 1, and supposing the only information we know is that 2 unit of $\boldsymbol{w_2}$ is better than 1 unit of $\boldsymbol{w_1}$, we then get the user preference $\boldsymbol{w_1} \leq \frac{2}{3}$ that reduces the state space $\boldsymbol{\mathcal{W}}$ as shown in Figure 1 (white background); in such context, we do not have a single dominating element for the set $\boldsymbol{A''}$, since for $\boldsymbol{0} \leq \boldsymbol{w_1} < \frac{1}{3}$ $(10,4)$ dominates $(4,7)$, and for $\frac{1}{3} < \boldsymbol{w_1} \leq \frac{2}{3}$ $(4,7)$ dominates $(10,4)$.

Minimality

Regarding question (1), we need to be able to eliminate unimportant choices, to make the list of options manageable, in particular, if we want to display the outcomes to the user. We interpret this as finding a minimal subset $\boldsymbol{A'}$ of $\boldsymbol{A}$ such that $Val_{\boldsymbol{A}}(\boldsymbol{w}) = Val_{\boldsymbol{A'}}(\boldsymbol{w})$ for every scenario $\boldsymbol{w} \in \boldsymbol{\mathcal{W}}$. For example, considering the set of choices $\boldsymbol{A}$ and the preference state space $\boldsymbol{\mathcal{W}}$ defined in Figure 1, the set $\boldsymbol{A'}$ is the minimal subset of $\boldsymbol{A}$.

Sets comparison

Question (2) concerns a case in which the user may have a choice between (I) being able to obtain any of the set of outcomes $\boldsymbol{A}$, and (II) any outcome in $\boldsymbol{B}$. For instance, $\boldsymbol{A}$ might correspond to hotels in Paris, and $\boldsymbol{B}$ to hotels in Lisbon, for a potential weekend away. We want to be able to determine if one of these clearly dominates the other; if, for instance, $\boldsymbol{A}$ dominates $\boldsymbol{B}$, then there may be no need for the system and the user to further consider $\boldsymbol{B}$, and, for example, may focus on Paris rather than Lisbon. We interpret this task as determining if in every scenario the utility $\boldsymbol{A}$ is at least that for $\boldsymbol{B}$, i.e., $Val_{\boldsymbol{A}}(\boldsymbol{w}) \geq Val_{\boldsymbol{B}}(\boldsymbol{w})$. In Figure 1 we can see a mathematical example where $\boldsymbol{A}$ dominates $\boldsymbol{B}$ in $\boldsymbol{\mathcal{W}}$ since $Val_{\boldsymbol{A}}(\boldsymbol{w}) \geq Val_{\boldsymbol{B}}(\boldsymbol{w})$ for any $\boldsymbol{w} \in \boldsymbol{\mathcal{W}}$.

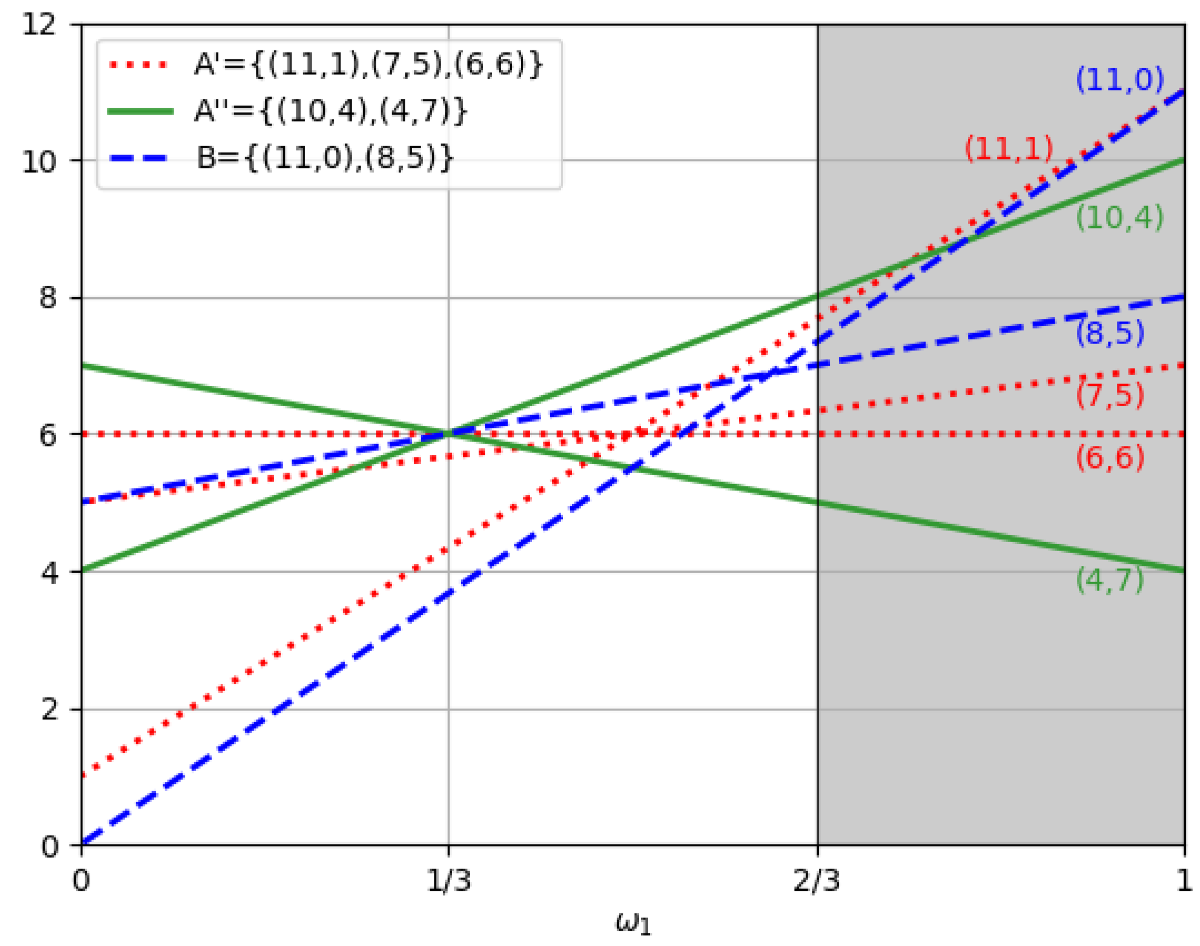

Figure 1: $f_w(\alpha) = w \cdot \alpha$ and $f_w(\beta) = w \cdot \beta$ for each $\alpha \in A$ and $\beta \in B$, where $A = A' \cup A'' = \{(11,1),(10,4),(7,5),(6,6),(4,7)\}$, $B = \{(11,0),(8,5)\}$ and $\mathcal{W} = \{w = (w_1, w_2) : w_1 + w_2 = 1 \;\&\; 0 \leq w_1 \leq \frac{2}{3}\}$ (white background).

Implementation and applications

In order to answer questions (1) and (2), we developed a linear programming method, and a method based on computing the extreme points of the epigraph of the value function (EEV). A possible application of our methods could be for example reducing the set of utility vectors derived for a multi-objective influence diagram [1].

Ongoing research

Using the main theoretical concepts of our work, we are developing an algorithm to compute the optimal subset $\boldsymbol{B}$ of cardinality $\boldsymbol{k}$ of an input set $\boldsymbol{A}$ of multi-attribute utility vectors with cardinality $\boldsymbol{n} > \boldsymbol{k}$. In this context, a subset $\boldsymbol{B}$ is optimal with respect to the partial user preferences given as input if it minimize the maximal regret, i.e. if $\boldsymbol{B}$ is the subset with the best worst-case scenario, where a scenario is a specific user preference (see [2] for more details).

References

R. Marinescu, A. Razak, and N. Wilson.
Multi-objective influence diagrams.
In *Proc. UAI-2012*, pages 574–583, 2012.

Paolo Viappiani and Craig Boutilier.
Regret-based optimal recommendation sets in conversational recommender systems.
In *Proceedings of the third ACM conference on Recommender systems*, pages 101–108. ACM, 2009.

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This work has been supported by a research grant by Science Foundation Ireland under grant number SFI/12/RC/2289

Ireland's European Structural and Investment Funds Programmes 2014-2020
Co-funded by the Irish Government and the European Union
EUROPEAN UNION
European Regional Development Fund
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