The small weight words of some Hermitian codes

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Cork, Workshop on Coding and Cryptography

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Acknowledgements

The work is jointly with Marco Pellegrini and our supervisor Massimiliano Sala.

The small weight words of some Hermitian codes

Hermitian curve

We consider the Hermitian curve χ over $\mathbb{F}_{q^2}$

$$x^{q+1} = y^q + y$$

The norm is a function $N : \mathbb{F}_{q^r} \to \mathbb{F}_q$ such that

$$N(x) = x^{1+q+\cdots+q^{r-1}}$$

The trace is a function $Tr : \mathbb{F}_{q^r} \to \mathbb{F}_q$ such that

$$Tr(x) = x + x^q + \cdots + x^{q^{r-1}}$$

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The Hermitian curve can be described as

$$N(x) = Tr(y), \quad \text{with } r = 2$$

This curve has exactly

$$n = (q^2 \text{ of } x) \cdot (q \text{ of } y) = q^3 \text{ rational points}$$

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Hermitian code

Definition

The evaluation map is

$$ev_{\mathcal{P}} : \mathbb{F}_q[x,y]/\langle x^{q+1} - y^q - y\rangle \longrightarrow (\mathbb{F}_{q^2})^n$$

$$ev_{\mathcal{P}}(f) = (f(P_1), \ldots, f(P_n))$$

Let m be a natural number, then we define

$$\mathcal{B}_{m,q} = \{x^r y^s \mid \mathrm{w}(x^r y^s) \leq m,\ 0 \leq s \leq q-1\}$$

So we consider

$$E_m = \langle ev_{\mathcal{P}}(f) \text{ such that } f \in \mathcal{B}_{m,q}\rangle$$

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Hermitian code

Therefore

$$C_m = (E_m)^{\perp} = \{\mathbf{c} \in (\mathbb{F}_q)^n \mid \mathbf{c} \cdot ev_{\mathcal{P}}(f) = 0 \text{ and } f \in \mathcal{B}_{m,q}\}$$

$C(m,q) = C_m$ is called Hermitian code. The parity-check matrix $\mathbf{H}$ of $C(m,q)$ is

$$\mathbf{H} = \begin{pmatrix} f_1(P_1) & \dots & f_1(P_n) \\ \vdots & \ddots & \vdots \\ f_i(P_1) & \dots & f_i(P_n) \end{pmatrix}$$

where $f_j \in \mathcal{B}_{m,q}$.

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The four phases of Hermitian codes

Phase	m
1	$0 \leq m \leq q^2 - q - 2$
2	$q^2 - q \leq m \leq 2q^2 - 2q - 2$
3	$2q^2 - 2q - 1 \leq m \leq q^3 - 1$
4	$q^3 \leq m < q^3 + q^2 - q - 1$

We have studied phase one, i.e. the case $d \leq q$.

The small weight words of some Hermitian codes

Corner Code

If **H** is composed of (the evaluation of) these sets

$$\begin{aligned} L_0^d &= \{1, x, \dots, x^{d-2}\} \\ L_1^d &= \{y, xy, \dots, x^{d-3}y\} \\ &\vdots \\ L_{d-2}^d &= \{y^{d-2}\} \end{aligned}$$

Then the code is called a Corner Code and it is indicated H_d^0.
The dimension of this code is

$$k = n - \frac{d(d-1)}{2}.$$

Edge Code

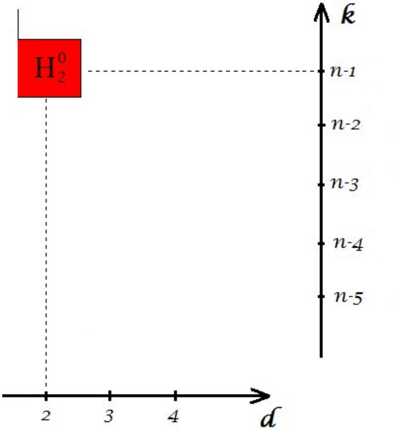
The code having parity-check matrix composed of $L_0^d \sqcup \cdots \sqcup L_{d-2}^d$ and of

$$\begin{aligned} l_1^d &= x^{d-1} \\ l_2^d &= x^{d-2}y \\ &\vdots \\ l_j^d &= x^{d-j}y^{j-1} \end{aligned}$$

is called an Edge Code, denoted with H_d^j $(1 \leq j \leq d-1)$.
The dimension of this code is

$$k = n - \frac{d(d-1)}{2} - j.$$

Corner Code e Edge Code



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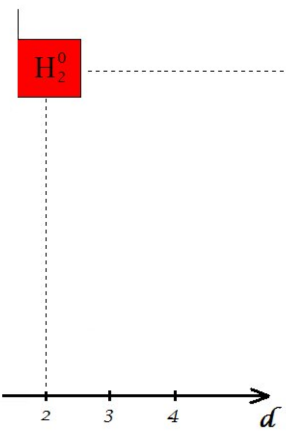

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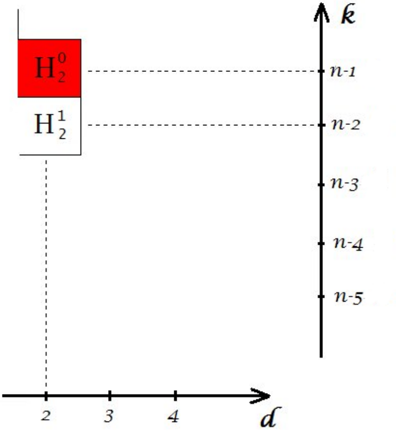

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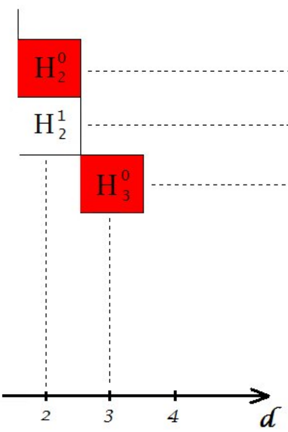

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- $\mathrm{H}_{\mathbf{3}}^{\mathbf{0}}$ is $[n, n-3, 3]$ code.
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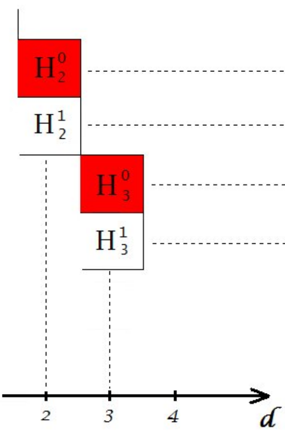
Corner Code e Edge Code



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- H_3^0 is $[n, n-3, 3]$ code. $\mathcal{B}_{m,q} = L_0^3 \sqcup L_1^3 = \{1, x, y\}$
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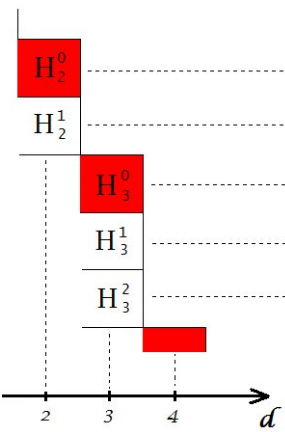

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- H_3^2 is $[n, n-5, 3]$ code. $\mathcal{B}_{m,q} = L_0^3 \sqcup L_1^3 \sqcup \{l_1^3, l_2^3\} = \{1, x, y, x^2, xy\}$

The number of codewords

Let $C(m,q) = \mathcal{C}_{m,q}$ be an Hermitian code. So

$$z \in \mathcal{C}_{m,q} \iff z\mathbf{H} = 0$$

$$\sum_{i=1}^{n} z_i f_j(P_i) = 0 \quad \forall j = 1,\ldots,n-k \text{ for } f_j \in \mathcal{B}_{m,q} = \{f_1,\ldots,f_{n-k}\}$$

All words of weight w correspond to solutions of this system:

$$J_{q,m}(w) = \begin{cases} \sum_{i=1}^{w} z_i x_i^r y_i^s = 0 \quad \forall\, r,s \quad \text{for} \quad x^r y^s \in \mathcal{B}_{m,q} \\ x_i^{q+1} - y_i^q - y_i = 0 \quad \forall\, i = 1,\ldots,w \\ x_i^{q^2} - x_i = 0 \quad y_i^{q^2} - y_i = 0 \quad z_i^{q^2-1} - 1 = 0 \quad \forall\, i = 1,\ldots,w \\ \prod_{1 \le i < j \le w}((x_i - x_j)^{q^2-1} - 1)((y_i - y_j)^{q^2-1} - 1) = 0 \end{cases}$$

$$A_w(\mathcal{C}_{m,q}(w)) = \frac{|\mathcal{V}(J_{q,m}(w))|}{w!}$$

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Minimum weight codewords for some Hermitian codes

Edge Code

Theorem

The minimum weight words of an Edge Code H_d^j are

$$A_d = q^2(q^2-1)\binom{q}{d}$$

We use this lemma

Lemma

The minimum weight words correspond to points lying on a vertical line.

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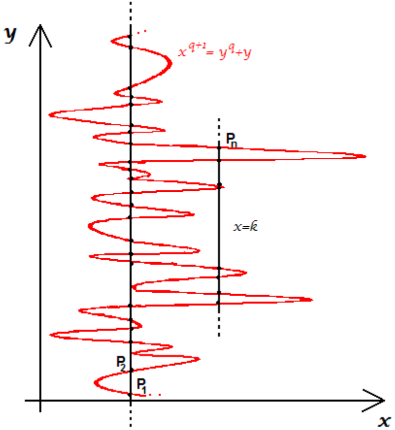

Let $x, y \in \mathbb{F}_{q^2}$

$$\begin{cases} x^{q+1} = y^q + y \\ x = k \end{cases}$$

We have to solve this equation

$$t = k^{q+1} = y^q + y$$

So

$$q^2 \quad \binom{q}{d} \quad \#z$$

$$\downarrow \qquad \downarrow$$

$$x \qquad y$$

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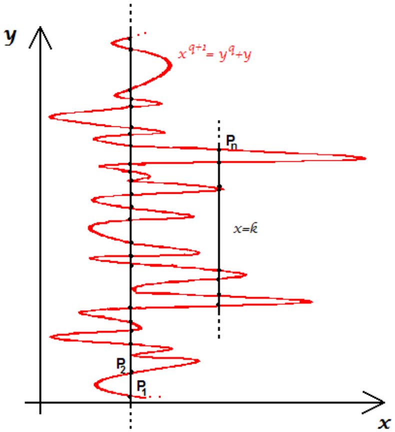

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$$\begin{matrix} q^2 & \binom{q}{d} & \#z \\ \downarrow & \downarrow & \\ x & y & \end{matrix}$$

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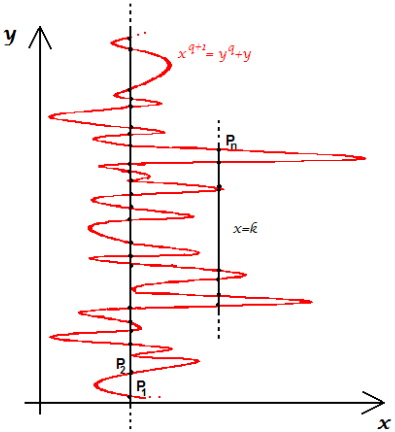

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We know that $\sum_{i=1}^{d} z_i x_i^r y_i^s = 0$ where $x^r y^s \in \mathcal{B}_{m,q}$. Then

$$\begin{cases} z_1 + \cdots + z_d = 0 \\ y_1 z_1 + \cdots + y_d z_d = 0 \\ \quad \vdots \\ y_1^{d-2} z_1 + \cdots + y_d^{d-2} z_d = 0 \end{cases}$$

The solutions in z are

$$(a_1\alpha, a_2\alpha, \ldots, a_{d-1}\alpha, \alpha) \quad \text{with} \quad \alpha \in \mathbb{F}_{q^2}^*.$$

For this reason

$$\#z = q^2 - 1.$$

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Corner Codes

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$$A_d = q^2(q^2-1)\binom{q}{d-1}\frac{q^3-d+1}{d}$$

To prove the theorem we use this

Lemma

The words of minimum weight are the points lying in the intersection of any line and the curve.

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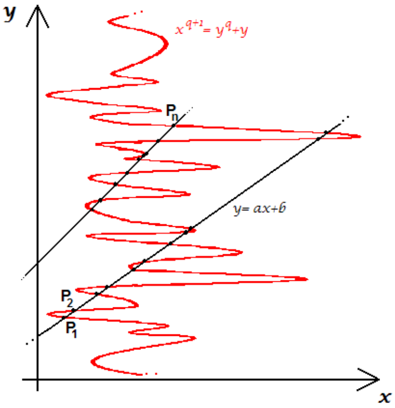

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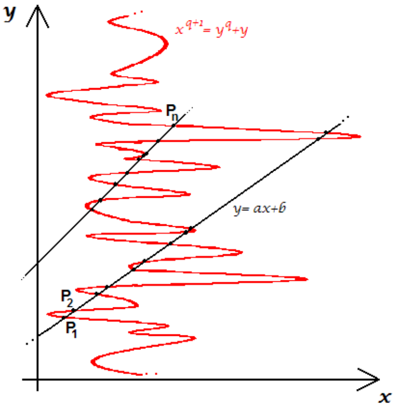
So

$$\underset{y}{\underset{\downarrow}{(q^4 - q^3)}} \quad \underset{x}{\underset{\downarrow}{\binom{q+1}{d}}} \quad \underset{z}{\underset{\downarrow}{(q^2 - 1)}}$$

Sketch of proof



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$$y \qquad x \qquad z$$

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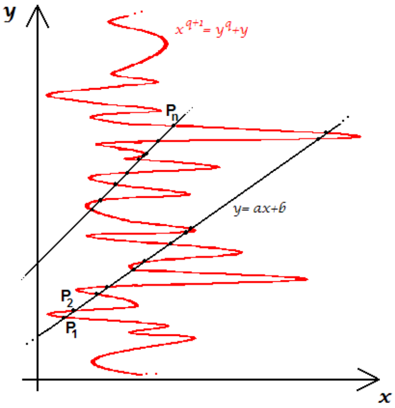

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whereas those corresponding to non-vertical lines are:

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So to find the result of the theorem we have to sum these two values.

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The small weight words of some Hermitian codes

The case with $y_1 = \cdots = y_{d+1}$

Theorem (Corner Code)

The number of words of weight $d+1$ with $y_1 = \cdots = y_{d+1}$ of a Corner Code H_d^0 is:

$$A_{d+1} = \frac{(q^2-q)(q^4-(d+1)q^2+d)}{(d+1)!}\binom{q+1}{d+1}$$

Theorem (Edge Code)

The words of weight $d+1$ with $y_1 = \cdots = y_{d+1}$ of an Edge Code H_d^j with $1 \leq j \leq d-1$ are:

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The case with $y_1 = \cdots = y_{d+1}$

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Sketch of proof for $y_1 = \ldots = y_{d+1}$

The points (x_i, y_i) lie on the Hermitian curve $x_i^{q+1} = y_i^q + y_i$.

$$\#y = q^2 - q \quad \text{and} \quad \#x = \binom{q+1}{d+1}$$

The z's have to verify $\sum_{i=1}^{d} z_i x_i^r y_i^s = 0$ where $x^r y^s \in \mathcal{B}_{m,q}$.

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$$\begin{cases} z_1 + \cdots + z_{d+1} = 0 \\ x_1 z_1 + \cdots + x_{d+1} z_{d+1} = 0 \\ x_1^2 z_1 + \cdots + x_{d+1}^2 z_{d+1} = 0 \\ \quad \vdots \\ x_1^{d-2} z_1 + \cdots + x_{d+1}^{d-2} z_{d+1} = 0 \end{cases}$$

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$$\Downarrow$$

$$\#z = q^2 \cdot q^2 - \#(z_i = 0 \text{ for some } i)$$

$$\#z = q^4 - (d+1)q + d$$

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$$\#z = q^2 - 1$$

The number of codewords of H_3^0

Coordinates of points | **H_3^0**

$x_i \neq x_j$	$y_i = y_j$	$\frac{1}{24}(q^4 - 4q^2 + 3)(q^2 - q)\binom{q+1}{4}$
	$y_1 = y_2 = y_3 \neq y_4$	0
	$y_1 = y_2 \neq y_3 = y_4$	$\frac{1}{8}q^2(q^2-1)^2(q-2)(q^3+2q^2-2q+1)$
	$y_1 = y_2 \neq y_3, y_4$ $y_3 \neq y_4$	$A_4 > \frac{1}{4}q^2(q^2-1)^2(2q^6-7q^5-6q^4+19q^3-9q^2-4q+4)$ $A_4 < \frac{1}{4}q^2(q^2-1)^2(2q^6-3q^5-11q^4+9q^3+25q^2-14q+4)$
	$y_i \neq y_j$	?
$x_i = x_j$	$y_i \neq y_j$	$\frac{1}{24}q^2(q^4-4q^2+3)\binom{q}{4}$

Thank you for your attention!