Mathematics Department, University College Cork
Twenty-Sixth Superbrain Examination, 2009

Dr. Anca Mustaţă, Dr. Ben McKay & Dr. Stephen Wills

Time allowed: *Three hours*
Tables and calculators may be used
Answer all questions

1. If x and y are positive integers and y is prime, solve the equation $x^2 = y^3 + 1$.

2. Evaluate the integral $\displaystyle\int_0^{1/2} 2\pi x \cot(\pi x)\, dx$.

3. Show how to assemble squares of side lengths $18, 15, 14, 10, 9, 8, 7, 4, 1$ to form a rectangle without leaving any gaps.

4. Compute $\displaystyle\sum_{n=1}^{\infty}\Big(\frac{1}{2n-1} - \frac{1}{2n}\Big)$.

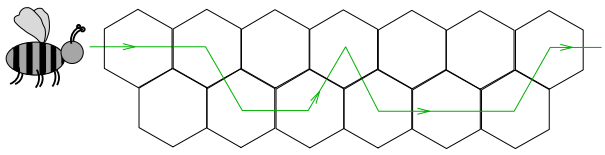
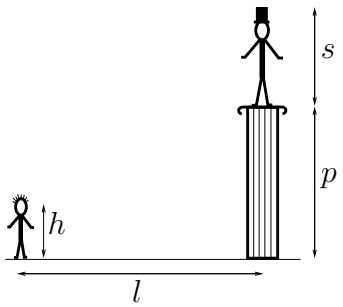
5. Let a, b, c be three distinct integers. Find all polynomials $P(x)$ with integer coefficients such that $P(a) = b$, $P(b) = c$, and $P(c) = a$.

6. Take a circle of radius 1 and a circle of radius 2, with centres 8 units apart. Pick a point p of the first circle, and a point q of the second. Let m be the midpoint of p and q. As p and q vary independently, what shape is described by the points m?

7. Inside a square of side length 1 are drawn a finite number of circles, with the sum of their circumferences being 10. Show that some straight line drawn across the square intersects at least four of the circles.

8. A coin is tossed repeatedly until either HH (two consecutive heads) or TH (tail then head) is observed. What is the probability that you stop tossing after seeing HH rather than TH?

9. A bee walks across the section of honeycomb illustrated, entering at the left and exiting at the right. Allowable moves at each stage are directly right, up and right, or down and right (where possible), but no move left can be taken. How many routes can the bee take? (An allowable path is shown.)



10. Whilst on holiday you go to observe a famous statue of height s on top of a pedestal of height p. Your eyes are a height h above the ground, where $h < p$. What distance l should you stand from the pedestal to ensure the best view? (Part of the question is to work out what is meant by the best view!)