

# INTERVARSITY MATHEMATICS COMPETITION 2008

NATIONAL UNIVERSITY OF IRELAND, MAYNOOTH

**Question 1.** Consider a regular  $n$ -gon inscribed inside a circle of radius 1. Each of the vertices of the  $n$ -gon lie on the circle. Draw a line between each pair of the  $n$  vertices. Prove that the sum of the squares of the lengths of these lines is  $n^2$ .

**Question 2.** There are 63 coins, all identical in appearance, and all identical in weight except one coin that is slightly heavier than the others. Three men wish to identify the heavy coin from the 63 coins using a peculiar set of pan scales. These pan scales have three pans placed symmetrically about a central pivot point. The only function of the scales is that if two pans each carry a load of equal weight, and the third pan carries a load of a different weight, then the movement of the scales indicates whether the different weight is heavier or lighter than the other two weights. Figure 1 is a diagram of the pan scales shown from above.

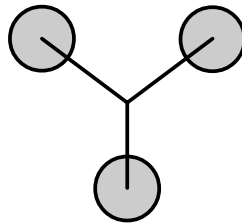


FIGURE 1

Each of the three men must perform one weighing of a selection of the coins without the other two men witnessing the result of the weighing. Beforehand, the three men plan a strategy, and after the weighings, the three men regroup and compare their results. Is there a strategy such that when the men regroup they can identify the heavy coin?

**Question 3.** Given positive numbers  $a_1 \geq a_2 \geq a_3$ ,  $b_1 \geq b_2 \geq b_3$ , and  $c_1 \geq c_2 \geq c_3$ , and three permutations  $\alpha$ ,  $\beta$ , and  $\gamma$  of the set  $\{1, 2, 3\}$ , prove that

$$a_1 b_1 c_1 + a_2 b_2 c_2 + a_3 b_3 c_3 \geq a_{\alpha(1)} b_{\beta(1)} c_{\gamma(1)} + a_{\alpha(2)} b_{\beta(2)} c_{\gamma(2)} + a_{\alpha(3)} b_{\beta(3)} c_{\gamma(3)}.$$

**Question 4.** Let  $(F(n))$  denote the usual Fibonacci sequence, which is defined by the properties  $F(1) = F(2) = 1$  and  $F(n+2) = F(n+1) + F(n)$  for  $n > 0$ . Let  $(a(n))$  denote the *iterated Fibonacci sequence*, which is given by the equation  $a(n) = F(F(n))$  for each  $n > 0$ . Prove that  $a(n)$  is a multiple of 144 whenever it is a multiple of 14.

**Question 5.** Take a positive integer  $N$  that ends in the digit 2. Move the 2 to the beginning of the number, and subtract 1 to get a new number,  $M$ . For example, if  $N = 52$ , then we move the 2 to the beginning of  $N$  to get 25, and then subtract 1 to obtain  $M = 24$ . What is the smallest value of  $N$  such that  $M = 2N$ ?

**Question 6.** For  $n = 1, 2, \dots$  let

$$a_n = n(n+1), \quad b_n = n+1.$$

What is the value of the convergent sequence

$$\frac{a_1}{b_1}, \quad \frac{a_1}{b_1 + \frac{a_2}{b_2}}, \quad \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3}}}, \dots \quad ?$$

**Question 7.** Let a triangle  $ABC$  have sides of length  $|AB| = 5$ ,  $|BC| = 6$ , and  $|CA| = 7$ , and let the lengths of the perpendiculars from a point  $P$  inside the triangle to the sides  $AB$ ,  $BC$ , and  $CA$  be  $x$ ,  $y$ , and  $z$ , respectively. Find the minimum value of the sum  $x^2 + y^2 + z^2$ .

**Question 8.** Let  $(a_n)$  be a sequence of non-negative integers such that

$$a_{n+2} = \binom{a_{n+1}}{a_n},$$

where

$$\binom{m}{k} = \begin{cases} \frac{m!}{k!(m-k)!} & \text{if } m \geq k, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that such a sequence tends to infinity if and only if  $2 \leq a_1 \leq a_2 - 2$ .

**Question 9.** Show that there does not exist a differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  which satisfies the following equation for all real values  $x$ ,

$$f'(x) \geq 1 + [f(x)]^2.$$

**Question 10.** In a room there are 2008 bulbs and 2008 buttons, and both of these are numbered from 1 to 2008. For each integer  $1 \leq i \leq 2008$ , pressing Button  $i$  changes the on or off status of Bulb  $i$  and one other bulb (the same other bulb each time). Assuming that all bulbs are initially off, prove that by pressing the appropriate combination of buttons we can turn at least 1340 of them simultaneously on. Prove also that if the effects of the buttons are chosen appropriately, 1340 is the largest number of bulbs that can be simultaneously on.