



Maths Intervarsities Competition 2000

10.00—13.00 March 4th

Dublin City University

Answer all questions.

1. Prove that

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}, \quad \text{for any positive integer } n.$$

2. Determine *all* positive integers n for which $2^n + 1$ is divisible by 3.

3. Show that the sequence

$$\sqrt{7}, \quad \sqrt{7 - \sqrt{7}}, \quad \sqrt{7 - \sqrt{7 + \sqrt{7}}}, \quad \sqrt{7 - \sqrt{7 + \sqrt{7 - \sqrt{7}}}}$$

converges and evaluate the limit.

4. The sequence $\{x_0, x_1, x_2, \dots\}$ is defined by the conditions

$$x_0 = a, \quad x_1 = b, \quad x_{n+1} = \frac{x_{n-1} + (2n-1)x_n}{2n}$$

Express $\lim_{n \rightarrow \infty} x_n$ concisely in terms of a and b (given numbers).

5. A random number generator can only select one of the nine integers $1, 2, \dots, 9$ and it makes these selections with equal probability. Determine the probability that after n selections the product of the n numbers will be divisible by 10.

6. Evaluate

$$\int_{-\infty}^{\infty} e^{-ax^2 - bx^{-2}} dx, \quad a, b > 0.$$

7. How many zeros are there at the end of the number $2000! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot 1999 \cdot 2000$?

8. It may seem odd but the sets $[0, 1]$ and $(0, 1)$ contain the same “number” of points. Find a one to one map of $[0, 1]$ onto $(0, 1)$.

9. The area T and an angle γ of a triangle are given. Determine the lengths of the sides a and b so that the side c , opposite to the angle γ , is as short as possible.

10. What is the smallest amount that may be invested at interest rate i , compounded annually, in order that one may withdraw £1 at the end of the first year, £4 at the end of the second year, \dots , £ n^2 at the end of the n -th year, \dots , in perpetuity?