Irish Intervarsity Mathematics Competition 1998.

Time allowed: *three hours*

1. Prove that there are just two ways to choose four distinct positive integers a, b, c, d, all less than 20 such that
$$ab = c^2 d, \quad bc = a^2.$$

2. If the perpendiculars from the vertices of a triangle to the respective opposite sides are produced to invert the circumcircle, prove that the segments of these perpendiculars between the orthocentre and the circumcircle are bisected by the sides of the triangle.

3. Divide the figure shown into 4 congruent regions.

4. Prove that it is impossible to make two loaded cubical dice in such a way that the sum of the two numbers shown when they are thrown is equally likely to take any of the values $2, 3, \ldots, 12$.

5. Form the equations whose roots are $\sin^2(\pi/7), \sin^2(2\pi/7), \sin^2(3\pi/7)$.
Hence or otherwise show that
$$\sum_{r=1}^{3} cosec^2(r\pi/7) = 8.$$

6. Show that $\int_0^{\pi/2} f(\cos\theta)d\theta = \int_0^{\pi/2} f(\sin\theta)d\theta$. Hence or otherwise prove that
$$\int_0^{\pi/2} \ln\sin\theta d\theta = -\frac{\pi}{2}\ln 2$$

7. Prove that
$$1998 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{1,000,000}} < 1999.$$

8. Show that there exists an infinite number of triples of positive integers (x, y, z) such that
$$x(x+1), \quad y(y+1), \quad z(z+1)$$
are in arithmetic progression.

9. Let $f : \mathbb{N} \to \mathbb{N}_0$ satisfy $f(1) = 0$ and
(i) $f(f(n)+2n-1) = n-1$, (ii) $f(f(n)+2n) = n$, (iii) $f(f(n)+2n+1) = n$.
Determine $f(1998)$.

10. Let X be a set with n elements, and $\mathcal{F}$ be a family of three-subsets of X such that any pair of subsets of $\mathcal{F}$ has at most one element in common. Show that there exists a subset M of X which has m elements, such that no member of $\mathcal{F}$ is a subset of M, and where
$$m(m+1) \geq 2n$$