On Linear-Programming Decoding of Nonbinary Expander Codes

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Literature Survey

- [Gallager ’62]
  Low-Density Parity-Check (LDPC) codes.
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- [Feldman et al. ’04] [Feldman Stein ’04] LP decoding on expander codes corrects a fraction of errors, achieves capacity.
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This Work

- LP decoding of nonbinary expander codes.
- The decoder corrects a number of errors which is approximately a quarter of a lower bound on the minimum distance.
- We consider:
  - Bipartite expander graph.
  - Two different types of constituent codes.
- The proof does not use a separate assumption on the symmetry of the LP polytope.
[Sipser Spielman ’95] [Barg Zémor ’01–’02]
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\[\text{Vitaly Skachek} \quad \text{LP Decoding of Nonbinary Expander Codes}\]
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C is a linear code of length $|E|$ over $\mathbb{F}$:

$$C = \left\{ c \in \mathbb{F}^{|E|} : \ (c)_{E(v)} \in C_A \text{ for every } v \in A \text{ and } (c)_{E(v)} \in C_B \text{ for every } v \in B \right\},$$
Code Construction

[Sipser Spielman ’95] [Barg Zémor ’01–’02]

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where $(c)_{E(v)}$ = the sub-word of $c$ that is indexed by the set of edges incident with $v$. 
Take $k = 2$, $\Delta = 3$, $n = 4$. Let $G_A$ and $G_B$ be generating matrices of $\mathcal{C}_A$ and $\mathcal{C}_B$ (respectively) over $\mathbb{F}_{2^2} = \{0, 1, \alpha, \alpha^2\}$:

$$
G_A = \begin{pmatrix}
1 & 1 & 1 \\
1 & \alpha & 0 \\
1 & \alpha & 0
\end{pmatrix},
$$

$$
G_B = \begin{pmatrix}
1 & 0 & 1 \\
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\end{pmatrix}.
$$
Example

Take $k = 2$, $\Delta = 3$, $n = 4$. Let $G_A$ and $G_B$ be generating matrices of $C_A$ and $C_B$ (respectively) over $\mathbb{F}_{2^2} = \{0, 1, \alpha, \alpha^2\}$:

$$G_A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha & 0 \end{pmatrix},$$

$$G_B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \alpha \end{pmatrix}.$$
Assume that all vertices in $\mathcal{G}$ have degree $\Delta$. The largest eigenvalue of the adjacency matrix $A_\mathcal{G}$ of $\mathcal{G}$ is $\Delta$. 
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Expander graph with

$$\lambda_G \leq 2\sqrt{\Delta} - 1$$

is called a *Ramanujan graph*. Constructions are due to [Lubotsky Philips Sarnak ’88], [Margulis ’88].
Parameters of Expander Codes

Code Rate

\[ R_C \geq r_A + r_B - 1. \]
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**Code Rate**

\[ R_C \geq r_A + r_B - 1. \]

**Relative Minimum Distance**

\[ \delta_C \geq \frac{\delta_A \delta_B - \gamma g \sqrt{\delta_A \delta_B}}{1 - \gamma g}. \]
General Notation

Let the codeword $c = \{c_e\}_{e \in E} \in \mathcal{C}$ be transmitted and the word $y = \{y_e\}_{e \in E} \in \mathbb{F}^{\lvert E \rvert}$ be received.
Let the codeword $c = (c_e)_{e \in E} \in \mathbb{C}$ be transmitted and the word $y = (y_e)_{e \in E} \in \mathbb{F}^{|E|}$ be received.

Define the mapping

$$\xi : \mathbb{F} \rightarrow \{0, 1\}^q \subset \mathbb{R}^q,$$

by

$$\xi(\alpha) = x = (x(\omega))_{\omega \in \mathbb{F}},$$

such that, for each $\omega \in \mathbb{F},$

$$x(\omega) = \begin{cases} 
1 & \text{if } \omega = \alpha \\
0 & \text{otherwise.}
\end{cases}$$
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Let \( \Xi(\mathbf{c}) = (\xi(c_{e_1}) \mid \xi(c_{e_2}) \mid \cdots \mid \xi(c_{|E|})) \).
For vectors $f \in \mathbb{R}^{q|E|}$, we adopt the notation

$$f = (f_{e_1} \mid f_{e_2} \mid \cdots \mid f_{e_{|E|}}),$$

where

$$\forall e \in E, \ f_e = (f^{(\alpha)}_e)_{\alpha \in \mathbb{F}}.$$
For vectors $f \in \mathbb{R}^{q|E|}$, we adopt the notation

$$f = (f_{e_1} | f_{e_2} | \cdots | f_{e_{|E|}}),$$

where

$$\forall e \in E, f_e = (f_e^{(\alpha)})_{\alpha \in \mathbb{F}}.$$

For all $e \in E, \alpha \in \mathbb{F}$, we use the variables $f_e^{(\alpha)} \geq 0$. 
Objective Function

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- For all $e \in E$, $\alpha \in \mathbb{F}$, we use the variables $f_e^{(\alpha)} \geq 0$.
- Variables $w_v, b$ for all $v \in V$ and all $b \in C(v)$: relative weights of local codewords $b$ associated with $E(v)$. 

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Variables $w_v, b$ for all $v \in V$ and all $b \in \mathcal{C}(v)$: relative weights of local codewords $b$ associated with $E(v)$.

The objective function is $\sum_{e \in E} \sum_{\alpha \in \mathbb{F}} \gamma_{e}^{(\alpha)} f_{e}^{(\alpha)}$, where $\gamma_{e}^{(\alpha)}$ is a function of the channel output.
Objective Function (cont.)

- For each $\alpha \in \mathbb{F}$ we set

$$
\gamma_e(\alpha) = \begin{cases} 
-1 & \text{if } \alpha = y_e \\
1 & \text{if } \alpha \neq y_e
\end{cases}.
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Let $f_e = \xi(\beta)$ for some $e \in E, \beta \in \mathbb{F}$. Then,

$$\sum_{\alpha \in \mathbb{F}} \gamma_e(\alpha) f_e(\alpha) = \begin{cases} 
-1 & \text{if } \beta = y_e \\
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\end{cases}.
$$

Suppose now that $f = \Xi(z)$ for some $z \in \mathbb{F}^{\mid E\mid}$. It follows that

$$
\sum_{e \in E} \sum_{\alpha \in \mathbb{F}} \gamma_e^{(\alpha)} f_e^{(\alpha)} + \mid E \mid = 2d(y, z),
$$

where $d(y, z)$ is the Hamming distance between $y$ and $z$. 
Maximize

\[ \sum_{e \in E, \alpha \in F} \left( -\gamma_e^{(\alpha)} \right) \cdot f_e^{(\alpha)} \]
Primal Problem

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subject to

\[ \forall v \in V : \sum_{b \in C(v)} w_{v,b} = 1 ; \]
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\[ \forall e = \{v, u\} \in E, \forall \alpha \in \mathbb{F} : f_{e}^{(\alpha)} = \sum_{b \in C(v)} : b_{e} = \alpha w_{v, b} ; \]

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\[ \forall e \in E, \alpha \in F : \quad f_e^{(\alpha)} \geq 0 ; \]

\[ \forall v \in V, b \in C(v) : \quad w_{v,b} \geq 0 . \]
Dual Witness Approach [Feldman et al. ’04]
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- There is a feasible combination of values of the variables $w_{v,b}$ that corresponds to $c$. 

Dual Witness

**Dual Witness Approach** [Feldman et al. ’04]

- The codeword \( c \in C \) was transmitted.
- There is a feasible combination of values of the variables \( w_v, b \) that corresponds to \( c \).

**Decoding Success**

The sufficient criteria for the decoding success is that this solution is the *unique* optimum of the primal LP decoding problem.
Decoding Success

Dual Polytope

Primal Polytope
Decoding Success

Dual Polytope

Min

Max

Primal Polytope
Decoding Success

**Dual Polytope**

**Feasible Point**

**Primal Polytope**
For each $\omega \in \mathbb{F}$, $e \in E$, and $v \in V$, such that $v$ is an endpoint of $e$, there is a variable $\tau_{v,e}^{(\omega)}$. 
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For each $v \in V$, there is a variable $\sigma_v$. 
Dual Problem

Minimize $\sum_{v \in V} \sigma_v$
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subject to $\forall e = \{v, u\} \in E, \forall \omega \in \mathbb{F}$ : $\tau_{v,e}^{(\omega)} + \tau_{u,e}^{(\omega)} \leq \gamma_{e}^{(\omega)}$;
Dual Problem

Minimize \( \sum_{v \in V} \sigma_v \)

subject to  
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\forall e = \{v, u\} \in E, \forall \omega \in \mathbb{F} : \quad \tau^{(\omega)}_{v,e} + \tau^{(\omega)}_{u,e} \leq \gamma^{(\omega)}_e ;
\]

\[
\forall v \in V, \forall b \in \mathcal{C}(v) : \quad \sum_{e \in E(v)} \tau^{(b_e)}_{v,e} + \sigma_v \geq 0 .
\]
Minimize \[ \sum_{v \in V} \nu_v \]

subject to \[ \forall e = \{v, u\} \in E, \forall \omega \in \mathbb{F} \setminus \{c_e\} : \quad \tau_{v,e}^{(\omega)} + \tau_{u,e}^{(\omega)} < \gamma_e^{(\omega)}; \]
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\[
\forall v \in V, \ \forall b \in \mathcal{C}(v) : \quad \sum_{e \in E(v)} \tau_{v,e}^{(b_e)} + \sigma_v \geq 0 .
\]
We aim at the objective value to be $|E| - 2d(y, c)$. This can be achieved by setting, for all $v \in V$, $\sigma_v = \frac{1}{2} \Delta - d((y)_{E(v)}, (c)_{E(v)})$. 
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<table>
<thead>
<tr>
<th>Condition</th>
<th>$\omega = c_e$</th>
<th>$\omega \neq c_e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_e$ is correct</td>
<td>$\tau_{v,e}^{(\omega)} = -\frac{1}{2}$</td>
<td>$\tau_{v,e}^{(\omega)} = \frac{1}{2} - \epsilon$</td>
</tr>
<tr>
<td>$y_e$ is in error</td>
<td>$\tau_{v,e}^{(\omega)} = \frac{1}{2}$</td>
<td>$\tau_{v,e}^{(\omega)} = -\frac{5}{2} - \epsilon$ or $\tau_{v,e}^{(\omega)} = \frac{3}{2}$ depends on the structure of the error</td>
</tr>
</tbody>
</table>
Uniqueness (again)

Minimize \[ \sum_{v \in V} \sigma_v \]

subject to
\[ \forall e = \{v, u\} \in E, \forall \omega \in F \setminus \{c_e\} : \quad \tau_{v,e}^{(\omega)} + \tau_{u,e}^{(\omega)} < \gamma_e^{(\omega)} ; \]
\[ \forall e = \{v, u\} \in E : \quad \tau_{v,e}^{(c_e)} + \tau_{u,e}^{(c_e)} \leq \gamma_e^{(c_e)} ; \]
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Here for all \( v \in V, \sigma_v = \frac{1}{2} \Delta - d((y)_{E(v)}, (c)_{E(v)}) \).
The assignment of the directions to the edges of the subgraph \( \mathcal{H} = (U_A \cup U_B, \mathcal{E}) \) is called a \((\rho_A, \rho_B)\)-orientation if each vertex \( v \in U_A \) and each vertex \( u \in U_B \) has at most \( \rho_A \Delta \) and \( \rho_B \Delta \) incoming edges in \( \mathcal{E} \), respectively.
Error Orientation

Definition

The assignment of the directions to the edges of the subgraph $\mathcal{H} = (U_A \cup U_B, \mathcal{E})$ is called a $(\rho_A, \rho_B)$-orientation if each vertex $v \in U_A$ and each vertex $u \in U_B$ has at most $\rho_A \Delta$ and $\rho_B \Delta$ incoming edges in $\mathcal{E}$, respectively.
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Error pattern orientation: for $\omega \neq c_e$ and $y_e$ in error, the value $\tau_{v,e}^{(\omega)} = -\frac{5}{2} - \epsilon$ will be assigned if the edge $e$ enters the vertex $v$. 

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LP Decoding of Nonbinary Expander Codes
Existence of \((< \frac{1}{4} \delta_A, < \frac{1}{4} \delta_B)\)-orientation yields a sufficiently small number of assignments \(-\frac{5}{2} - \epsilon\). This, in turn, yields that

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\sum_{e \in E(v)} \tau_{v,e}^{(b_e)} \geq -\sigma_v .
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\]

**Lemma**

Let \(\mathcal{H} = (U_A \cup U_B, \mathcal{E})\) be a subgraph of \(\mathcal{G} = (A \cup B, E)\). Assume that \(|\mathcal{E}| \leq (\alpha \beta - \frac{1}{2} \gamma_{\mathcal{G}})\Delta n\) for some \(\alpha, \beta \in (0, 1]\), such that \(\gamma_{\mathcal{G}} \leq \sqrt{\alpha \beta}\), and \(\frac{1}{2} \alpha \Delta, \frac{1}{2} \beta \Delta\) are both integers. Then, \(\mathcal{E}\) contains a \((\beta/2, \alpha/2)\)-orientation.
Theorem

- Let $\theta_A > 0$ ($\theta_B > 0$) be the largest number such that $\theta_A < \delta_A$ ($\theta_B < \delta_B$) and $\frac{1}{4}\theta_A\Delta$ ($\frac{1}{4}\theta_B\Delta$, respectively) is integer.
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- Let $C$ be as above, and assume that $\gamma_G \leq \frac{1}{2}\sqrt{\theta_A \theta_B}$.
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Let $C$ be as above, and assume that $\gamma_G \leq \frac{1}{2} \sqrt{\theta_A \theta_B}$.

Then, the LP decoder corrects any error pattern of a size less than or equal to $(\frac{1}{4} \theta_A \theta_B - \frac{1}{2} \gamma_G) \Delta n$. 
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Remark

It is possible to improve (slightly) on the low-order term in the expression for the number of correctable errors in the statement of the main result. In the present work, we omit this analysis.