Weight Distributions of Doubly-Generalized LDPC Codes

Mark F. Flanagan
University College Dublin, Ireland

Enrico Paolini, Marco Chiani
DEIS, University of Bologna, Italy

Marc Fossorier
ETIS ENSEA / UCP / CNRS UMR-8051, France

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Low-density parity-check codes (LDPC codes) [Gallager ’62]

- Check Nodes
- Variable Nodes

CNs may be viewed as single parity check (SPC) codes
VNs may be viewed as repetition (REP) codes
Background

- Low-density parity-check codes (LDPC codes) [Gallager '62]
- **Tanner graph**: check nodes (CNs) and variable nodes (VNs) connected by edges
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Bits associated with VNs; CNs represent parity checks
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Sometimes called a *generalized* CN
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Represent a promising solution for low-rate channel coding schemes, due to an overall rate loss introduced by the generalized CNs
Doubly-Generalized LDPC codes (D-GLDPC Codes)

[Sipser Spielman ’96] [Dolinar ’03] [Wang Fossorier ’06]
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- Further generalization of GLDPC concept
- Facilitates much greater design flexibility in terms of code rate, e.g. high-rate codes may be constructed
D-GLDPC Example

(7, 4) Hamming codes

Π

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REP code

SPC code

(7, 4) Hamming codes
(Selected) Previous Work on Growth Rate

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- For the general case, we present a **polynomial-system** solution for the growth rate.
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- For \( t \in I_v \), denote by \( k_t, q_t \) and \( p_t \) the VN dimension, length and minimum distance, respectively
We define the polynomials

\[ \rho(x) = \sum_{t \in I_c} \rho_t x^{s_t-1} ; \quad \lambda(x) = \sum_{t \in I_v} \lambda_t x^{q_t-1} \]
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E = \frac{n}{\int \lambda} \text{ Tanner graph edges, and } m = E \int \rho \text{ CNs}
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The D-GLDPC code consists of

\[ N = \frac{n}{\int \lambda} \sum_{t \in I_v} \frac{\lambda_t k_t}{q_t} \] bits and \( M = \frac{m}{\int \rho} \sum_{t \in I_c} \frac{\rho_t(s_t - h_t)}{s_t} \) checks
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A member of the ensemble corresponds to a permutation on the \( E \) edges connecting CNs to VNs.
The weight enumerating polynomial for CN type $t \in I_c$ is given by

$$A^{(t)}(x) = \sum_{u=0}^{s_t} A^{(t)}_u x^u = 1 + \sum_{u=r_t}^{s_t} A^{(t)}_u x^u.$$
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- The bivariate weight enumerating polynomial for VN type \( t \in I_v \) is given by

\[
B^{(t)}(x, y) = \sum_{u=0}^{k_t} \sum_{v=0}^{q_t} B^{(t)}_{u,v} x^u y^v = 1 + \sum_{u=1}^{k_t} \sum_{v=p_t}^{q_t} B^{(t)}_{u,v} x^u y^v .
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- Here $B^{(t)}_{u,v} \geq 0$ denotes the number of weight-$v$ codewords generated by input words of weight $u$, for VNs of type $t$. 

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Further Definitions (CN side)

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We also define for each \( t \in I_c \)

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\[ \mathbf{p = 2} : \text{For each } t \in X_v, \text{ define the (nonempty) set} \]

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- **p = 2**: Define the polynomial $P(x)$ by
  $$P(x) = \sum_{t \in X_v} \lambda_t \sum_{i \in L_t} \frac{2B_{i,2}^{(t)}}{q_t} x^i.$$
Remarks

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- All coeffs of $P(x)$ positive $\implies P(x)$ monotonically increasing on $[0, \infty)$ $\implies$ its inverse $P^{-1}(x)$ is well-defined and unique on $[0, \infty)$. 

Case $r = p = 2$: both $C$ and the polynomial $P(x)$ depend only on the CNs and VNs with minimum distance equal to 2.
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Growth Rate

Definition

The growth rate of the weight distribution of the irregular D-GLDPC ensemble sequence \( \{ \mathcal{M}_n \} \) is defined by

\[
G(\alpha) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_{\mathcal{M}_n} [N_{\alpha n}]
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- The limit assumes the inclusion of only those positive integers \( n \) for which \( \alpha n \in \mathbb{Z} \) and \( \mathbb{E}_{\mathcal{M}_n} [N_{\alpha n}] \) is positive (i.e. where the expression whose limit we seek is well defined)
Case I: $\alpha \rightarrow 0 \ (r = p = 2)$

**Theorem**

Suppose $r = p = 2$. For sufficiently small $\alpha$, the growth rate of the weight distribution is given by

$$G(\alpha) = \alpha \log \left[ \frac{1}{P^{-1}(1/C)} \right] + O(\alpha^2).$$
The stability condition yields an upper bound on the iterative decoding threshold $q^*$ of a code ensemble over the BEC.

The parameter $1 - \frac{1}{C} \left( \frac{1}{C} \right)$ is a direct generalization of the corresponding parameter for LDPC and GLDPC codes.
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This bound was derived for irregular LDPC, GLDPC and D-GLDPC codes.
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Case I: $\alpha \to 0$ (general)

**Theorem**

For sufficiently small $\alpha$, the growth rate of the weight distribution is given by

$$G(\alpha) = \left( \frac{T}{\psi} \right) \alpha \log \alpha + \alpha \left[ \log \left( \frac{1}{P_1^{-1}(1)} \right) + \frac{T}{\psi} \log \left( \frac{1}{P_2(P_1^{-1}(1))} \right) \right] + O(\alpha^2).$$

Here $\psi = r/(r-1)$, and $T \geq 0$ with equality if and only if $r = p = 2$. Also $P_1(x)$ and $P_2(x)$ are polynomials with positive coefficients.
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Mark F. Flanagan  
Weight Distributions of D-GLDPC Codes
Case I: $\alpha \to 0$ (general)

Theorem

For sufficiently small $\alpha$, the growth rate of the weight distribution is given by

$$G(\alpha) = \left( \frac{T}{\psi} \right) \alpha \log \alpha + \alpha \left[ \log \left( \frac{1}{P_1^{-1}(1)} \right) \right.$$

$$+ \frac{T}{\psi} \log \left( \frac{1}{P_2(P_1^{-1}(1))} \right) \left. \right] + O(\alpha^2).$$

- Here $\psi = r/(r - 1)$, and $T \geq 0$ with equality if and only if $r = p = 2$.
- Also $P_1(x)$ and $P_2(x)$ are polynomials with positive coefficients.
Case II: arbitrary $\alpha$

Let $x_0, y_0, z_0$ and $\beta$ be the unique positive real solutions to the 4 $\times$ 4 system of polynomial equations

\[
z_0 \left( \int \frac{\rho}{\lambda} \right) \sum_{s \in I_c} \gamma_s \frac{dA(s)}{dz}(z_0) = \beta,
\]

\[
x_0 \sum_{t \in I_v} \delta_t \frac{\partial B(t)}{\partial x}(x_0, y_0) = \alpha,
\]

\[
y_0 \sum_{t \in I_v} \delta_t \frac{\partial B(t)}{\partial y}(x_0, y_0) = \beta,
\]

and

\[
\left( \beta \int \lambda \right) (1 + y_0z_0) = y_0z_0.
\]
Case II: arbitrary $\alpha$

**Theorem**

The growth rate of the weight distribution of the irregular D-GLDPC ensemble sequence $\{M_n\}$ is given by

$$G(\alpha) = \sum_{t \in I_v} \delta_t \log B^{(t)}(x_0, y_0) - \alpha \log x_0$$

$$+ \left( \frac{\int \rho}{\int \lambda} \right) \sum_{s \in I_c} \gamma_s \log A^{(s)}(z_0) + \log \left( \frac{1 - \beta \int \lambda}{\int \lambda} \right)$$
We have derived an expression for the asymptotic growth rate of irregular doubly-generalized LDPC code ensembles.
Conclusion

- We have derived an expression for the asymptotic growth rate of irregular doubly-generalized LDPC code ensembles
  - $\alpha \to 0$; compact analytical result
  - General $\alpha$; polynomial system solution
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  - General $\alpha$; polynomial system solution
- This generalizes known results for LDPC and GLDPC codes
- Identifies the parameter $\frac{1}{P-1}(1/C)$ as playing a key role in D-GLDPC code ensembles
- Extends the link between growth rate and stability condition over the BEC to the case of D-GLDPC codes
- Allows for exact numerical evaluation of the ensemble relative minimum distance.
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